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Strongly Focused Gravitational Waves

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Abstract: Recently, Christodoulou [Chr] proved that trapped spheres can form in evolution, through the focusing of gravitational waves, from a generic initial state. His work is the motivation for the present paper, in which we consider the same physical problem, using very different mathematical methods. Our approach is based on a controlled "far field expansion". By a systematic use of scaling symmetries, we regularize Christodoulou's singular "short pulse method", rigorously track vacuum solutions by the far field expansion and exhibit trapped spheres that first appear deep inside the far field region. Our presentation is self-contained. In the final section, we present a detailled outline of the construction of another, more subtle, expansion that allows us to continue the solutions beyond the far field region to within any fixed "finite distance" from the (expected) singularity. From a methodological perspective, the underlying aim of this paper is the development of a general method for constructing solutions to the vacuum Einstein equations by controlled expansions.

1. Introduction

Formal and controlled (perturbation) expansions are common tools in mathematics and physics. In general relativity, see, for example, [AnRe], [BBM], [Cha]. This paper is a first step in the development of a hybrid method, combining formal expansions and simple tools from the theory of hyperbolic partial differential equations (such as energy estimates), to construct generic classical solutions to the vacuum Einstein equations, for a wide variety of well posed problems with natural small parameters. The purpose of this paper is to illustrate the methodology, by carrying it out in all detail for a concrete, physically interesting situation.

Christodoulou [Chr] showed that strongly focused gravitational waves, coming in from past null infinity, generate trapped spheres. One of the most important innovations of [Chr] is the introduction of a small parameter δ and a picture in which δ represents the duration of a spherical pulse traveling along a null hypersurface. The amplitude of the pulse is scaled so that, roughly speaking, the total incoming energy per unit advanced time is proportional to $\frac{1}{\delta}$. Christodoulou refers to this picture as his "short pulse method". This physical setup triggered our interest in this problem.

To illustrate our approach, we shall construct strongly focused gravitational wave solutions to the vacuum Einstein equations by means of controlled expansions that exhibit trapped spheres.

In Christodoulou's picture, the limit $\delta \downarrow 0$ is singular. An important preliminary step in our analysis is to identify a one parameter group, indexed by $\mathfrak{A} \neq 0$, of (anisotropic) scaling transformations (see, Section 3) that regularizes the limit $\delta = \mathfrak{A}^4 \downarrow 0$. We work almost exclusively in this regularized picture. Our construction of strongly focused gravitational wave solutions in the regularized picture is conceptually and technically independent from [Chr]. In fact, our formalism is so different, that any discussion of the relation to [Chr] is deferred until Section 9 (see, Remark 9.3). We use:

- A reformulation of the vacuum Einstein equations in terms of an $\mathbb{R}^{31}$ valued field Φ (rather than a metric g), a first order quasilinear, symmetric hyperbolic system **(SHS)** for Φ, a $\mathbb{R}^{32}$ valued field $\Phi^\sharp$ of constraints constructed out of Φ, and a "dual" linear, homogeneous symmetric hyperbolic system $\widehat{\textbf{(SHS)}}$ for $\Phi^\sharp$. Namely, for each field Φ satisfying **(SHS)** and $\Phi^\sharp = 0$, one may canonically construct a Ricci flat, Lorentzian manifold, and conversely, there is a sufficiently small neighborhood of any point on any Ricci flat, Lorentzian manifold that arises in this way. The dual system $\widehat{\textbf{(SHS)}}$ is an intermediary tool that is used to show that the constraint field $\Phi^\sharp$ vanishes under suitable conditions.
- A unique formal power series solution $[\,\Phi\,]$, in a natural small parameter, to an appropriate well posed initial value problem for **(SHS)** that also satisfies the constraint equation $[\,\Phi^\sharp\,] = 0$. Here, $[\,\Phi^\sharp\,]$ is the formal power series constraint field constructed out of $[\,\Phi\,]$.
- A standard, local existence theorem and standard energy estimates for **(SHS)** that are used to show that, under suitable (generic) conditions, the formal solution $[\,\Phi\,]$ is an asymptotic expansion for a unique classical solution Φ to the same initial value problem for **(SHS)** . Energy estimates for $\widehat{\textbf{(SHS)}}$ are used to show that $\Phi^\sharp = 0$.

We emphasize that:

- The reformulation of the Einstein equations in terms of Φ, $\Phi^\sharp$, **(SHS)** , $\widehat{\textbf{(SHS)}}$ has very useful analytic (first order symmetric hyperbolic systems) and algebraic (all the nonlinearities are quadratic polynomials) properties.
- In our approach, geometry only appears in the formulation of the problem, and in the interpretation of the final results. This is in sharp contrast to [Chr] and [ChrKl], in which the very character of the method is consistently geometrical.

We have tried to present our construction in a transparent form with as many details as possible, so that it is accessible to the general reader without any specific background in general relativity or hyperbolic partial differential equations. For example, we include the derivation of the single (L^2) Sobolev inequality that is used. In fact, the discussion, up to the formation of trapped spheres, is entirely self contained, apart from the reference to [Tay] for a simple local existence theorem for quasilinear symmetric hyperbolic systems defined on the product of a time interval with a torus. For these reasons, this paper is longer than it might be. We have, however, omitted lengthy, but straight forward, direct verifications (typical of general relativity) of many equations and algebraic identities. We have included an index of notation (Appendix A).

We conclude this introduction with an overview of the contents of this paper. Section 2 gives completely self-contained but unmotivated statements about the purely algebraic reformulation of the vacuum Einstein equations. This reformulation is motivated and derived in three appendices:

- Appendix B introduces the underlying general formalism. It is based on work of Friedrich [Fr] and the Newman-Penrose complex tetrad formalism [NP].
- In Appendices C and D we choose a gauge, adapted to the problem we are solving, which forces the "angular derivatives" in **(SHS)** to appear only in very special places, with the consequence that the equations are easier to control.

The reader may be put off by the multi-page equations of Section 2. However, he or she should not be discouraged, because later, in Propositions 5.2, 5.3, 5.4, we will derive relevant / irrelevant forms of these equations, that exhibit the essential constituents that have to be treated carefully, and sweep everything else into "generic terms" that we don't need to know much about. We will almost exclusively work with this simple, transparent form of the equations.

Section 3 introduces a number of exact symmetry transformations of the equations of Section 2. In particular, the global anisotropic scaling $\mathfrak{A}$ (Definition 3.5) plays a central role for everything we do.

In Section 4, we define a two-parameter family of fields $\mathcal{M}_{a,\mathfrak{A}}$, solving **(SHS)** and the constraint equations. For all parameter values $a, \mathfrak{A} \neq 0$, the field $\mathcal{M}_{a,\mathfrak{A}}$ corresponds to Minkowski space. It will be the background for our far field expansion (see below). $\mathcal{M}_{a,\mathfrak{A}}$ is obtained from $\mathcal{M}_{1,1}$ by applying the scaling symmetries of Section 3.

In Section 5, an (asymptotic) characteristic initial value problem for the focusing of gravitational waves is informally stated. It is motivated by [Chr].

A formal solution $[\mathit{\Phi}]$ to the characteristic initial value problem is constructed in Section 6. To exhibit trapped spheres, it is sufficient to expand directly around past null infinity. That is, we make a far (weak) field (power series) expansion in $\frac{1}{u}$, morally the distance to past null infinity. The expansion is very simple, it can be written down explicitly, and one sees trapped spheres directly in the lowest order term. Somewhat surprisingly, trapped spheres appear deep inside the formal radius of convergence of the far field expansion, that is very close to past null infinity.

In Section 7, we reduce local existence for **(SHS)** to a standard result from [Tay]. We then develop energy estimates that enable us to show, in Sections 8 and 9, that, under suitable (generic) conditions, every formal solution $[\mathit{\Phi}]$ is an asymptotic expansion for a unique classical solution $\mathit{\Phi}$ with $\mathit{\Phi}^\sharp = 0$, which exists at least up to a small, fixed fraction of the formal radius of convergence of $[\mathit{\Phi}]$, and contains trapped spheres.

The formation of a trapped sphere is important, but not an end in itself. Two more pressing questions about strongly focused gravitational wave solutions are:

- Does a horizon form, under suitable conditions?
- How do these solutions behave as they are continued beyond the "far field region"?

To see that a horizon forms in strongly focused gravitational wave solutions, one would first have to continue them to include all of "future null infinity". We take a step in this direction by showing, in Section 9, that, under suitable conditions, our solutions become arbitrarily close to the Schwarzschild solution on a strip that extends from the location of the trapped sphere all the way out to past null infinity. As soon as a sufficiently well controlled expansion around the Schwarzschild / Kerr family becomes available, one could conclude that a horizon has formed.

In Subsection 9.5 we sketch a method for continuing the solutions out of the "far field region" using a more powerful expansion in $\mathfrak{A}$. The expansion in $\mathfrak{A}$ around the regular limit $\mathfrak{A} \downarrow 0$ is both more fundamental and more subtle. It is an expansion around a two parameter family of decoupled, fully nonlinear two dimensional systems. Fortunately, these two dimensional systems can be solved "quasi explicitly". Higher order terms

in $\mathfrak{A}$ can be constructed and controlled. In the far field region the expansion in $\frac{1}{u}$ can be rearranged into the $\mathfrak{A}$ expansion, and vice versa. We follow the $\mathfrak{A}$ expansion into the strong field regime, as far as it will take us (see, the figure in Subsection 9.5). The rigorous control of the $\mathfrak{A}$ expansion and its consequences will be reported on in all detail in another place. It is a second example of our methodology.

We appreciate the great effort that Demetrios Christodoulou invested over many years to nurture the mathematical study of general relativity at ETH Zurich.

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2. A Reformulation of the Vacuum Einstein Equations

Our method for constructing solutions to the vacuum Einstein equations has two parts. The first is algebraic, the second analytic. Here, we present the purely algebraic part. It is a reformulation of the vacuum Einstein equations that is carefully tailored to the constructive analytic tools used in the second, purely analytic part.

This section compresses the intuition and logic of the three leisurely Appendices B, C and D into elementary, but very lengthy, totally unmotivated and, to the contemporary eye, unsightly, definitions and statements.

Let

$$\big(x^1 \, , \, x^2 \, , \, x^3 \, , \, x^4 \big) \;=\; \big(\xi^1 \, , \xi^2 \, , \, \underline{u} \, , \, u \big)$$

be a coordinate system on the open subset $\mathcal{U}$ of $\mathbb{R}^4$, and

$$\varPhi(x) \;=\; \big(\varPhi_1(x) \, , \, \varPhi_2(x) \, , \, \varPhi_3(x) \big) \;=\; \big(e(x) \, , \, \gamma(x) \, , \, w(x) \big)$$

any sufficiently differentiable field on $\mathcal{U}$ taking values in

$$\mathcal{R} = \big\{ (e, \gamma, w) \in \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 \;\big|\; e_3, \, e_4, \, e_5, \, \gamma_2, \, \gamma_6 \in \mathbb{R} \, \big\}, \tag{2.1}$$

a real vector space of dimension 31. Throughout this paper, $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$.

Remark 2.1. Later on, the complex coordinate $\xi = \xi^1 + i\xi^2$ will play the role of an "angular" coordinate. See, for instance, Remark 4.1.

Definition 2.1. *To any sufficiently differentiable field $\varPhi : \mathcal{U} \to \mathcal{R}$, satisfying the conditions*

$$\begin{aligned} (\star) \quad &: \quad e_3 > 0 \\ (\star\,\star) \quad &: \quad \Im\big(e_1 \, \overline{e}_2\big) \neq 0 \end{aligned}$$

at every point of $\mathcal{U}$, we associate three fields $\mathbf{F}_a{}^\mu$, $\boldsymbol{\Gamma}_{ajk}$, $\mathbf{W}_{abjk}$ and a complex frame $F_a = \mathbf{F}_a{}^\mu \frac{\partial}{\partial x^\mu}$ on $\mathcal{U}$. Here and below, small Latin and small Greek indices run from one to four. The fields are uniquely determined by:

- $\boldsymbol{\Gamma}_{ajk} = -\boldsymbol{\Gamma}_{akj}$ *and* $\mathbf{W}_{abjk} = -\mathbf{W}_{abkj}$ *and* $\mathbf{W}_{abjk} = -\mathbf{W}_{bajk}$.

•

$$\big(\,\mathbf{F}_a{}^{\mu}\,\big) \;=\; \begin{pmatrix} e_1 & e_2 & 0 & 0 \\ \overline{e}_1 & \overline{e}_2 & 0 & 0 \\ e_4 & e_5 & 0 & 1 \\ 0 & 0 & e_3 & 0 \end{pmatrix}$$

$$\big(\,\boldsymbol{\Gamma}_{a(jk)}\,\big) \;=\; \begin{pmatrix} \gamma_3 + \overline{\gamma}_4 & \overline{\gamma}_7 & \gamma_6 & \gamma_1 & \gamma_2 & \gamma_3 - \overline{\gamma}_4 \\ -\gamma_4 - \overline{\gamma}_3 & \gamma_6 & \gamma_7 & \gamma_2 & \overline{\gamma}_1 & -\gamma_4 + \overline{\gamma}_3 \\ \gamma_8 - \overline{\gamma}_8 & 0 & 0 & -\gamma_3 + \overline{\gamma}_4 & \gamma_4 - \overline{\gamma}_3 & \gamma_8 + \overline{\gamma}_8 \\ 0 & \overline{\gamma}_5 & \gamma_5 & 0 & 0 & 0 \end{pmatrix}$$

$$\big(\,\mathbf{W}_{(ab)(jk)}\,\big) \;=\; \begin{pmatrix} w_3 + \overline{w}_3 & \overline{w}_4 & -w_4 & w_2 & -\overline{w}_2 & w_3 - \overline{w}_3 \\ \overline{w}_4 & \overline{w}_5 & 0 & 0 & -\overline{w}_3 & -\overline{w}_4 \\ -w_4 & 0 & w_5 & -w_3 & 0 & -w_4 \\ w_2 & 0 & -w_3 & w_1 & 0 & w_2 \\ -\overline{w}_2 & -\overline{w}_3 & 0 & 0 & \overline{w}_1 & \overline{w}_2 \\ w_3 - \overline{w}_3 & -\overline{w}_4 & -w_4 & w_2 & \overline{w}_2 & w_3 + \overline{w}_3 \end{pmatrix}$$

The matrix indices (ab), (jk) run over the ordered sequence

$$(12) \quad (31) \quad (32) \quad (41) \quad (42) \quad (34)$$

The complex frame is written as:

- $\big(F_1, F_2, F_3, F_4\big) = \big(D, \overline{D}, N, L\big)$ *or, equivalently,*

$$D \;=\; e_1\, \tfrac{\partial}{\partial \xi^1} + e_2\, \tfrac{\partial}{\partial \xi^2} \quad , \quad N = e_4\, \tfrac{\partial}{\partial \xi^1} + e_5\, \tfrac{\partial}{\partial \xi^2} + \tfrac{\partial}{\partial u} \quad , \quad L = e_3\, \tfrac{\partial}{\partial \underline{u}} \tag{2.2}$$

The vector fields N and L are always real.

Proposition 2.1. *The field $\mathbf{W}_{abjk}$ has the symmetries*

$$\begin{aligned} \mathbf{W}_{abjk} &= -\mathbf{W}_{bajk} & \qquad \mathbf{W}_{ajk\ell} + \mathbf{W}_{a\ell jk} + \mathbf{W}_{ak\ell j} &= 0 \\ \mathbf{W}_{abjk} &= \mathbf{W}_{jkab} & \mathbf{g}^{aj}\, \mathbf{W}_{abjk} &= 0 \end{aligned}$$

where the matrix $\mathbf{g}_{ab}$ and its inverse $\mathbf{g}^{ab}$ are given by

$$\big(\,\mathbf{g}_{ab}\,\big) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \big(\,\mathbf{g}^{ab}\,\big) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.3}$$

That is, $\mathbf{W}_{abjk}$ has the algebraic properties of a Weyl field.

Proof. By direct inspection. □

Remark 2.2. Later on it will be important to drop $(\star\star)$. That is, to allow the frame to collapse.

The next definition singles out an important class of fields $\Phi(x)$ and relates them to Ricci-flat Lorentzian manifolds, that is, vacuum spacetimes.

Definition 2.2. *A field $\Phi : \mathcal{U} \to \mathcal{R}$ is a* **vacuum field** *when:*

- *Conditions* $(\star)$ *and* $(\star\star)$ *are all satisfied at every point of* $\mathcal{U}$.
- *The Levi-Civita connection for the complex linear metric* $\mathbf{g}$ *on* $\mathcal{U}$ *is given by*

$$\mathbf{g}\big(\nabla_{F_a}F_j,\,F_k\big)=\boldsymbol{\Gamma}_{ajk}$$

where $\mathbf{g}(F_a,F_b)\overset{def}{=}\mathbf{g}_{ab}$ *(see,* (2.3)*) and*

$$\mathbf{g}\big(\nabla_{F_a}F_j,\,F_k\big)\ \overset{def}{=}\ \tfrac{1}{2}\Big(-\mathbf{g}\big(F_a,[F_j,F_k]\big)+\mathbf{g}\big(F_k,[F_a,F_j]\big)+\mathbf{g}\big(F_j,[F_k,F_a]\big)\Big)$$

- *The Riemann tensor for the Levi-Civita connection is given by*

$$\mathbf{R}_{abjk}\ \overset{def}{=}\ \mathbf{g}\Big(\big[\nabla_{F_j},\nabla_{F_k}\big]F_b-\nabla_{[F_j,F_k]}F_b,\,F_a\Big)\ =\ \mathbf{W}_{abjk}$$

Consequently, the Ricci curvature vanishes, since $\mathbf{W}_{abjk}$ *is traceless.*
- *The coordinate functions* $\underline{u}$ *and* u *are both solutions to the eikonal equation. More precisely,* e_3N *and* L *are null geodesic vector fields that are minus the gradients of* $\underline{u}$ *and* u.

Remark 2.3. For any $\mathcal{R}$-valued field Φ satisfying the conditions $(\star)$ and $(\star\star)$ the metric given by $\mathbf{g}(F_a,F_b)=\mathbf{g}_{ab}$ is real in the sense that $\mathbf{g}(X,Y)$ is real whenever X and Y are real vector fields. Over the reals, it has signature $(-,+,+,+)$. Bear in mind that $\mathbf{g}_{ab}$ are the components with respect to the *complex* frame F_a. If Φ is a vacuum field, then $(\mathcal{U},\mathbf{g})$ is a Ricci-flat Lorentzian manifold, that is, a solution to the vacuum Einstein equations.

Remark 2.4. It is natural to ask whether all Ricci-flat Lorentzian manifolds arise, at least locally, in this way. The answer to this question is given in Appendices C and D.

Remark 2.5. Assume $\Phi(x)$ is a vacuum field. Declare $L+N$ to be future directed. Let $S_{\underline{u},u}$ be the intersection of the level sets of $\underline{u}$ and u, which by Definition 2.2 are null hypersurfaces. The traces of the future-directed second fundamental forms of $S_{\underline{u},u}$ relative to the level sets of u and $\underline{u}$ are given by $\mathbf{g}(\nabla_D L,\overline{D})+\mathbf{g}(\nabla_{\overline{D}}L,D)=2\gamma_2$ and $\mathbf{g}(\nabla_D N,\overline{D})+\mathbf{g}(\nabla_{\overline{D}}N,D)=2\gamma_6$. By (2.1), they are real, as they should be. By definition, $S_{\underline{u},u}$ is a *trapped surface* when γ_2 and γ_6 are strictly negative everywhere on $S_{\underline{u},u}$. Equivalently, $S_{\underline{u},u}$ is trapped if an infinitesimal shift of $S_{\underline{u},u}$ along either L or N (both future-directed null vector fields, orthogonal to $S_{\underline{u},u}$) induces a pointwise decrease of the area element.
The basic examples of (closed) trapped surfaces in a vacuum spacetime are the spherical $\mathrm{SO}(3)$ orbits inside the horizon of a Schwarzschild spacetime. Closed trapped surfaces appear in the the formulation of Penrose's incompleteness theorem, see [Pen].

We need a criterion for a field to be a vacuum field. To this end, we make two more definitions.

Definition 2.3. *Suppose, condition* $(\star)$ *is satisfied at every point of the domain* $\mathcal{U}$. *Let* $\Phi(x)=\big(e(x),\,\gamma(x),\,w(x)\big):\mathcal{U}\to\mathcal{R}$ *be a sufficiently differentiable field, and let the weights* $\lambda_1,\,\lambda_2,\,\lambda_3,\,\lambda_4$ *be strictly positive functions on* $\mathcal{U}$. *The* **Quasilinear Symmetric Hyperbolic System (SHS)** *for the field* $\Phi(x)$ *is*

$$\mathbf{A}(\Phi)\,\Phi\ =\ \mathbf{f}(\Phi) \tag{2.4}$$

Here, $\mathbf{A}(\Phi) = \mathbf{A}_1(\Phi) \oplus \mathbf{A}_2(\Phi) \oplus \mathbf{A}_3(\Phi)$ *is the first order, matrix differential operator, with coefficients that are affine linear functions (over* $\mathbb{R}$*) of* Φ*, given by*

$$\mathbf{A}_1(\Phi) \;=\; \operatorname{diag}\big(L,\, L,\, N,\, L,\, L\big) \tag{2.5a}$$

$$\mathbf{A}_2(\Phi) \;=\; \operatorname{diag}\big(L,\, L,\, L,\, L,\, N,\, N,\, N,\, L\big) \tag{2.5b}$$

$$\mathbf{A}_3(\Phi) \;=\; \begin{pmatrix} \lambda_1 N & \lambda_1 D & 0 & 0 & 0 \\ \lambda_1 \overline{D} & \lambda_1 L + \lambda_2 N & \lambda_2 D & 0 & 0 \\ 0 & \lambda_2 \overline{D} & \lambda_2 L + \lambda_3 N & \lambda_3 D & 0 \\ 0 & 0 & \lambda_3 \overline{D} & \lambda_3 L + \lambda_4 N & \lambda_4 D \\ 0 & 0 & 0 & \lambda_4 \overline{D} & \lambda_4 L \end{pmatrix} \tag{2.5c}$$

Observe that the "angular" operators D, $\overline{D}$ *only appear in* $\mathbf{A}_3(\Phi)$*. Also,*

$$\mathbf{f}(\Phi) \;=\; \mathbf{f}_1(\Phi) \oplus \mathbf{f}_2(\Phi) \oplus \mathbf{f}_3(\Phi) \;=\; \begin{pmatrix} \mathbf{f}_{11} \\ \vdots \\ \mathbf{f}_{51} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{f}_{12} \\ \vdots \\ \mathbf{f}_{82} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{f}_{13} \\ \vdots \\ \mathbf{f}_{53} \end{pmatrix}$$

is the quadratically nonlinear vector valued function given by

$$\mathbf{f}_{j1} = \begin{cases} -\, e_1\gamma_2 - \overline{e}_1\gamma_1 & \\ -\, e_2\gamma_2 - \overline{e}_2\gamma_1 & \\ e_3\; 2\,\Re\,\gamma_8 & \star \\ 2\,\Re\,\big(-\, e_1\gamma_4 + e_1\gamma_5 + e_1\overline{\gamma}_3\big) & \star \\ 2\,\Re\,\big(-\, e_2\gamma_4 + e_2\gamma_5 + e_2\overline{\gamma}_3\big) & \star \end{cases}$$

$$\mathbf{f}_{j2} = \begin{cases} -\, 2\gamma_1\gamma_2 - w_1 & \\ -\, |\gamma_1|^2 - \gamma_2\gamma_2 & \star \\ +\, \gamma_1\gamma_4 - \gamma_1\gamma_5 - \gamma_2\gamma_3 - w_2 & \\ -\, \gamma_2\gamma_4 + \gamma_2\gamma_5 + \gamma_3\overline{\gamma}_1 & \\ -\, \gamma_3\gamma_7 + \gamma_4\gamma_6 - \gamma_5\gamma_6 - \gamma_5\gamma_8 + \gamma_5\overline{\gamma}_8 - \gamma_6\overline{\gamma}_3 + \gamma_7\overline{\gamma}_4 - \gamma_7\overline{\gamma}_5 + w_4 & \\ -\, \gamma_6\gamma_6 - \gamma_6\; 2\,\Re\,\gamma_8 - |\gamma_7|^2 & \star \\ -\, 2\gamma_6\gamma_7 - 3\gamma_7\gamma_8 + \gamma_7\overline{\gamma}_8 - w_5 & \\ -\, 2\gamma_3\gamma_4 + 2\gamma_3\gamma_5 + \gamma_3\overline{\gamma}_3 + \gamma_4\overline{\gamma}_4 - \gamma_4\overline{\gamma}_5 - \gamma_5\overline{\gamma}_4 + w_3 & \end{cases}$$

$$\mathbf{f}_{j3} = \begin{cases} -\, \lambda_1\big(3\gamma_1 w_3 - 6\gamma_3 w_2 + \gamma_6 w_1 - 4\gamma_8 w_1 + 4\overline{\gamma}_4 w_2\big) \\ -\, \lambda_1\big(4\gamma_2 w_2 + 4\gamma_4 w_1 + \gamma_5 w_1\big) \\ \qquad\qquad -\, \lambda_2\big(2\gamma_1 w_4 - 3\gamma_3 w_3 + 2\gamma_6 w_2 - 2\gamma_8 w_2 + 3\overline{\gamma}_4 w_3\big) \\ -\, \lambda_2\big(3\gamma_2 w_3 + 2\gamma_4 w_2 + 2\gamma_5 w_2 + \gamma_7 w_1\big) - \lambda_3\big(\gamma_1 w_5 + 3\gamma_6 w_3 + 2\overline{\gamma}_4 w_4\big) \\ -\, \lambda_3\big(2\gamma_2 w_4 + 3\gamma_5 w_3 + 2\gamma_7 w_2\big) - \lambda_4\big(3\gamma_3 w_5 + 4\gamma_6 w_4 + 2\gamma_8 w_4 + \overline{\gamma}_4 w_5\big) \\ -\, \lambda_4\big(\gamma_2 w_5 - 2\gamma_4 w_4 + 4\gamma_5 w_4 + 3\gamma_7 w_3\big) \end{cases}$$

Definition 2.4. *Let $\Phi(x) = \big(e(x), \gamma(x), w(x)\big) : \mathcal{U} \to \mathcal{R}$ be a sufficiently differentiable field, and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be strictly positive weight functions on $\mathcal{U}$. The associated* **constraint field**

$$\Phi^\sharp(x) \;=\; \big(\Phi^\sharp_1(x),\, \Phi^\sharp_2(x),\, \Phi^\sharp_3(x)\big) \;=\; \big(t(x),\, u(x),\, v(x)\big) \;=\; \begin{pmatrix} t_1 \\ \vdots \\ t_5 \end{pmatrix} \oplus \begin{pmatrix} u_1 \\ \vdots \\ u_9 \end{pmatrix} \oplus \begin{pmatrix} v_1 \\ \vdots \\ v_3 \end{pmatrix}$$

on $\mathcal{U}$ taking values in

$$\widehat{\mathcal{R}} = \big\{(t,u,v) \in \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3 \;\big|\; t_1,\, t_2 \in \mathbb{R}\,\big\} \tag{2.6}$$

is given by

$$t_j = \begin{cases} -\,2\,\Im\,\big(D(\overline{e}_1) + \overline{e}_1\gamma_3 + \overline{e}_1\overline{\gamma}_4\big) \\ -\,2\,\Im\,\big(D(\overline{e}_2) + \overline{e}_2\gamma_3 + \overline{e}_2\overline{\gamma}_4\big) \\ D(e_3) - e_3\gamma_3 + e_3\overline{\gamma}_4 + e_3\overline{\gamma}_5 \\ \overline{D}(e_4) - N(\overline{e}_1) - e_1\gamma_7 - \overline{e}_1\gamma_6 + \overline{e}_1\overline{\gamma}_8 - \overline{e}_1\gamma_8 \\ \overline{D}(e_5) - N(\overline{e}_2) - e_2\gamma_7 - \overline{e}_2\gamma_6 + \overline{e}_2\overline{\gamma}_8 - \overline{e}_2\gamma_8 \end{cases} \tag{2.7a}$$

$$u_j = \begin{cases} D(\gamma_2) - \overline{D}(\gamma_1) + w_2 - 3\gamma_1\gamma_4 - \gamma_1\overline{\gamma}_3 - \gamma_2\gamma_3 + \gamma_2\overline{\gamma}_4 \\ D(\gamma_4) + \overline{D}(\gamma_3) - w_3 + 2\gamma_3\gamma_4 + \gamma_1\gamma_7 - \gamma_2\gamma_6 + \gamma_3\overline{\gamma}_3 + \gamma_4\overline{\gamma}_4 \\ \overline{D}(\gamma_3) - \overline{D}(\overline{\gamma}_4) + N(\gamma_2) \\ \qquad - w_3 + 2\gamma_3\gamma_4 - 2\gamma_4\overline{\gamma}_4 + \gamma_1\gamma_7 + \gamma_2\gamma_6 - \gamma_2\gamma_8 - \gamma_2\overline{\gamma}_8 \\ D(\overline{\gamma}_4) - D(\gamma_3) - N(\gamma_1) \\ \qquad + 3\gamma_1\gamma_8 + 2\gamma_3\gamma_3 - 2\gamma_3\overline{\gamma}_4 - \gamma_1\gamma_6 - \gamma_1\overline{\gamma}_8 - \gamma_2\overline{\gamma}_7 \\ \overline{D}(\gamma_5) - L(\gamma_7) - \gamma_2\gamma_7 - \gamma_4\gamma_5 + \gamma_5\gamma_5 - \gamma_5\overline{\gamma}_3 - \gamma_6\overline{\gamma}_1 \\ L(\gamma_6) - D(\gamma_5) - w_3 + \gamma_1\gamma_7 + \gamma_2\gamma_6 - \gamma_3\gamma_5 - \gamma_5\overline{\gamma}_4 - \gamma_5\overline{\gamma}_5 \\ D(\gamma_8) - N(\gamma_3) - 2\gamma_3\gamma_6 + \gamma_3\gamma_8 - \gamma_3\overline{\gamma}_8 + \gamma_4\overline{\gamma}_7 + \gamma_6\overline{\gamma}_4 \\ N(\gamma_4) + \overline{D}(\gamma_8) + w_4 - 2\gamma_3\gamma_7 + \gamma_4\gamma_6 + \gamma_4\gamma_8 - \gamma_4\overline{\gamma}_8 + \gamma_7\overline{\gamma}_4 \\ D(\gamma_7) - \overline{D}(\gamma_6) - w_4 + 3\gamma_3\gamma_7 + \gamma_4\gamma_6 - \gamma_6\overline{\gamma}_3 + \gamma_7\overline{\gamma}_4 \end{cases} \tag{2.7b}$$

$$v_j = \begin{cases} \lambda_1\big(\,\overline{D}(w_1) + L(w_2) + 4\gamma_2 w_2 + 4\gamma_4 w_1 + \gamma_5 w_1\big) \\ \lambda_2\big(\,\overline{D}(w_2) + L(w_3) + 3\gamma_2 w_3 + 2\gamma_4 w_2 + 2\gamma_5 w_2 + \gamma_7 w_1\big) \\ \lambda_3\big(\,\overline{D}(w_3) + L(w_4) + 2\gamma_2 w_4 + 3\gamma_5 w_3 + 2\gamma_7 w_2\big) \end{cases} \tag{2.7c}$$

Proposition 2.2. *Suppose, conditions $(\star)$ and $(\star\star)$ are all satisfied at every point of $\mathcal{U}$. Then, the field $\Phi(x)$ is a vacuum field if and only if there exist strictly positive weight functions λ_1, λ_2, λ_3, λ_4 on $\mathcal{U}$ such that $\Phi(x)$ is a solution to the quasilinear symmetric hyperbolic system* **(SHS)** *and the constraint field $\Phi^\sharp(x) = 0$ everywhere on $\mathcal{U}$.*

Proof. Follows from the Appendices B, C, D. See, Propositions B.1, D.1, D.2, D.3. □

Remark 2.6. Proposition 2.2, together with Definition 2.2, is a reformulation of the vacuum Einstein equations. When does the solution Φ to a well posed problem for **(SHS)** also satisfy $\Phi^\sharp = 0$?

Proposition 2.3. *Suppose, that condition $(\star)$ is satisfied and there are strictly positive weight functions λ_1, λ_2, λ_3, λ_4 on $\mathcal{U}$, such that $\Phi(x)$ is a C^2 solution to* **(SHS)** *. Then, the constraint field $\Phi^\sharp$ is a classical solution to the "dual"* **Homogeneous, Linear (over $\mathbb{R}$) Symmetric Hyperbolic System $\widehat{\textbf{(SHS)}}$** *:*

$$\widehat{\mathbf{A}}(\Phi)\,\Phi^\sharp \;=\; \widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\,\Phi^\sharp$$

In particular, if the data for any well posed problem for the system $\widehat{\textbf{(SHS)}}$ vanishes, then the constraint field $\Phi^\sharp(x)$ vanishes everywhere. Here, $\widehat{\mathbf{A}}(\Phi) = \widehat{\mathbf{A}}_1 \oplus \widehat{\mathbf{A}}_2 \oplus \widehat{\mathbf{A}}_3$ is the first order, matrix differential operator

$$\widehat{\mathbf{A}}_1 \;=\; \operatorname{diag}\big(L,\, L,\, N,\, L,\, L\big) \tag{2.8a}$$

$$\widehat{\mathbf{A}}_2 \;=\; \operatorname{diag}\big(L,\, L,\, L,\, L,\, N,\, N,\, L,\, L,\, N\big) \tag{2.8b}$$

$$\widehat{\mathbf{A}}_3 \;=\; \begin{pmatrix} \frac{1}{\lambda_1}N + \frac{1}{\lambda_2}L & \frac{1}{\lambda_2}D & 0 \\ \frac{1}{\lambda_2}\overline{D} & \frac{1}{\lambda_2}N + \frac{1}{\lambda_3}L & \frac{1}{\lambda_3}D \\ 0 & \frac{1}{\lambda_3}\overline{D} & \frac{1}{\lambda_3}N + \frac{1}{\lambda_4}L \end{pmatrix} \tag{2.8c}$$

and $\widehat{\mathbf{f}}(\Phi, \partial_x\Phi)$ is a linear (over $\mathbb{R}$) transformation acting on $\Phi^\sharp = (t, u, v)$:

$$\widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\,\Phi^\sharp \;=\; \big(\widehat{\mathbf{f}}_1 \oplus \widehat{\mathbf{f}}_2 \oplus \widehat{\mathbf{f}}_3\big)\Phi^\sharp \;=\; \begin{pmatrix} \widehat{\mathbf{f}}_{11} \\ \vdots \\ \widehat{\mathbf{f}}_{51} \end{pmatrix} \Phi^\sharp \oplus \begin{pmatrix} \widehat{\mathbf{f}}_{12} \\ \vdots \\ \widehat{\mathbf{f}}_{92} \end{pmatrix} \Phi^\sharp \oplus \begin{pmatrix} \widehat{\mathbf{f}}_{13} \\ \vdots \\ \widehat{\mathbf{f}}_{33} \end{pmatrix} \Phi^\sharp$$

where Φ has been suppressed on the right hand side, and

$$\widehat{\mathbf{f}}_{j1}\,\Phi^\sharp = \begin{cases} -\,2\gamma_2 t_1 + 2\,\Im\left(t_3 \frac{\partial \overline{e}_1}{\partial \underline{u}}\right) + 2\,\Im\left(\overline{e}_1 u_1\right) \\ -\,2\gamma_2 t_2 + 2\,\Im\left(t_3 \frac{\partial \overline{e}_2}{\partial \underline{u}}\right) + 2\,\Im\left(\overline{e}_2 u_1\right) \\ (-\gamma_6 + 2\gamma_8)t_3 - \overline{\gamma}_7 \overline{t}_3 - \overline{t}_4 \frac{\partial e_3}{\partial \xi^1} - \overline{t}_5 \frac{\partial e_3}{\partial \xi^2} + e_3 u_7 + e_3 \overline{u}_8 \\ i(\overline{\gamma}_3 - \gamma_4 + \gamma_5)t_1 - \gamma_2 t_4 - \overline{\gamma}_1 \overline{t}_4 - \overline{t}_3 \frac{\partial e_4}{\partial \underline{u}} + \overline{e}_1 u_3 - e_1 \overline{u}_4 + e_1 u_5 - \overline{e}_1 \overline{u}_6 \\ i(\overline{\gamma}_3 - \gamma_4 + \gamma_5)t_2 - \gamma_2 t_5 - \overline{\gamma}_1 \overline{t}_5 - \overline{t}_3 \frac{\partial e_5}{\partial \underline{u}} + \overline{e}_2 u_3 - e_2 \overline{u}_4 + e_2 u_5 - \overline{e}_2 \overline{u}_6 \end{cases}$$

$$\widehat{\mathbf{f}}_{j2}\,\Phi^{\sharp} =$$

$$\begin{cases} -t_3\frac{\partial\gamma_2}{\partial\underline{u}}+\overline{t}_3\frac{\partial\gamma_1}{\partial\underline{u}}-3\gamma_2u_1+\gamma_1\overline{u}_1+\frac{1}{\lambda_1}v_1 \\ -t_3\frac{\partial\gamma_4}{\partial\underline{u}}-\overline{t}_3\frac{\partial\gamma_3}{\partial\underline{u}}-\gamma_4u_1+\gamma_5u_1-\gamma_3\overline{u}_1-2\gamma_2u_2-\gamma_1u_5-\gamma_2u_6-\frac{1}{\lambda_2}v_2 \\ -\overline{t}_3\frac{\partial\gamma_3}{\partial\underline{u}}+\overline{t}_3\frac{\partial\overline{\gamma}_4}{\partial\underline{u}}+\overline{\gamma}_3u_1-\gamma_4u_1+\gamma_5u_1 \\ \qquad\qquad -2\gamma_2u_3+\overline{\gamma}_1u_4+\gamma_1\overline{u}_4-\gamma_1u_5+\gamma_2\overline{u}_6-\frac{1}{\lambda_2}v_2 \\ +t_3\frac{\partial\gamma_3}{\partial\underline{u}}-t_3\frac{\partial\overline{\gamma}_4}{\partial\underline{u}}+(\gamma_3-\overline{\gamma}_4+\overline{\gamma}_5)u_1+\gamma_1u_3+\gamma_1\overline{u}_3-2\gamma_2u_4+\gamma_2\overline{u}_5-\gamma_1u_6 \\ -t_4\frac{\partial\gamma_5}{\partial\xi^1}-t_5\frac{\partial\gamma_5}{\partial\xi^2}-\gamma_7u_3+\gamma_6\overline{u}_4-2\gamma_6u_5-2\gamma_8u_5+2\overline{\gamma}_8u_5 \\ \qquad\qquad +\gamma_7u_6+\gamma_7\overline{u}_6+\gamma_5\overline{u}_7-\gamma_5u_8-\gamma_4u_9+\gamma_5u_9+\overline{\gamma}_3u_9 \\ +\overline{t}_4\frac{\partial\gamma_5}{\partial\xi^1}+\overline{t}_5\frac{\partial\gamma_5}{\partial\xi^2}+\gamma_6\overline{u}_3-\gamma_7u_4+\overline{\gamma}_7u_5+\gamma_7\overline{u}_5 \\ \qquad\qquad -2\gamma_6u_6+\gamma_5u_7-\gamma_5\overline{u}_8+\gamma_3u_9-\overline{\gamma}_4u_9+\overline{\gamma}_5u_9+\frac{1}{\lambda_3}v_2 \\ -t_3\frac{\partial\gamma_8}{\partial\underline{u}}-\gamma_3u_2+\overline{\gamma}_4u_2-\overline{\gamma}_5u_2+\gamma_3\overline{u}_3+\gamma_4u_4-\gamma_5u_4 \\ \qquad\qquad -\gamma_4\overline{u}_5-2\gamma_3u_6+\overline{\gamma}_4u_6-\gamma_2u_7-\gamma_1u_8-\frac{1}{\lambda_2}v_1 \\ -\overline{t}_3\frac{\partial\gamma_8}{\partial\underline{u}}+\overline{\gamma}_3u_2-\gamma_4u_2+\gamma_5u_2-\gamma_4u_3+\gamma_5u_3-\gamma_3\overline{u}_4 \\ \qquad\qquad +2\gamma_3u_5-\overline{\gamma}_4u_5+\gamma_4\overline{u}_6-\overline{\gamma}_1u_7-\gamma_2u_8+\frac{1}{\lambda_3}v_3 \\ +t_4\frac{\partial\gamma_6}{\partial\xi^1}-\overline{t}_4\frac{\partial\gamma_7}{\partial\xi^1}+t_5\frac{\partial\gamma_6}{\partial\xi^2}-\overline{t}_5\frac{\partial\gamma_7}{\partial\xi^2}-3\gamma_7u_7+\gamma_6\overline{u}_7 \\ \qquad\qquad +\gamma_6u_8+\gamma_7\overline{u}_8-3\gamma_6u_9-2\gamma_8u_9+\gamma_7\overline{u}_9+\frac{1}{\lambda_4}v_3 \end{cases}$$

$$\begin{aligned} \widehat{\mathbf{f}}_{13}\,\Phi^{\sharp} = &-it_1\frac{\partial w_2}{\partial\xi^1}-it_2\frac{\partial w_2}{\partial\xi^2}+t_3\frac{\partial w_3}{\partial\underline{u}}-t_4\frac{\partial w_1}{\partial\xi^1}-t_5\frac{\partial w_1}{\partial\xi^2} \\ &+3w_3u_1+2w_2u_2+4w_2u_3-2w_2u_6+4w_1u_8+w_1u_9 \\ &+\big[-\tfrac{4}{\lambda_2}\gamma_2-\tfrac{2}{\lambda_1}\gamma_6+\tfrac{3}{\lambda_1}\gamma_8+\tfrac{1}{\lambda_1}\overline{\gamma}_8+(\tfrac{1}{\lambda_1})^2N(\lambda_1)+(\tfrac{1}{\lambda_2})^2L(\lambda_2)\big]v_1 \\ &+\big[(\tfrac{1}{\lambda_2})^2D(\lambda_2)+\tfrac{4}{\lambda_2}\gamma_3-\tfrac{4}{\lambda_2}\overline{\gamma}_4-\tfrac{1}{\lambda_2}\overline{\gamma}_5\big]v_2-\tfrac{2}{\lambda_3}\gamma_1v_3 \\ \widehat{\mathbf{f}}_{23}\,\Phi^{\sharp} = &-it_1\frac{\partial w_3}{\partial\xi^1}-it_2\frac{\partial w_3}{\partial\xi^2}+t_3\frac{\partial w_4}{\partial\underline{u}}-t_4\frac{\partial w_2}{\partial\xi^1}-t_5\frac{\partial w_2}{\partial\xi^2} \\ &+2w_4u_1+3w_3u_3-3w_3u_6+2w_2u_8+2w_2u_9 \\ &+\big[-\tfrac{2}{\lambda_2}\gamma_4-\tfrac{2}{\lambda_2}\gamma_5+(\tfrac{1}{\lambda_2})^2\overline{D}(\lambda_2)\big]v_1 \\ &+\big[-\tfrac{3}{\lambda_3}\gamma_2-\tfrac{3}{\lambda_2}\gamma_6+\tfrac{1}{\lambda_2}\gamma_8+\tfrac{1}{\lambda_2}\overline{\gamma}_8+(\tfrac{1}{\lambda_2})^2N(\lambda_2)+(\tfrac{1}{\lambda_3})^2L(\lambda_3)\big]v_2 \\ &+\big[\tfrac{1}{\lambda_3}\gamma_3-\tfrac{3}{\lambda_3}\overline{\gamma}_4-\tfrac{1}{\lambda_3}\overline{\gamma}_5+(\tfrac{1}{\lambda_3})^2D(\lambda_3)\big]v_3 \\ \widehat{\mathbf{f}}_{33}\,\Phi^{\sharp} = &-it_1\frac{\partial w_4}{\partial\xi^1}-it_2\frac{\partial w_4}{\partial\xi^2}+t_3\frac{\partial w_5}{\partial\underline{u}}-t_4\frac{\partial w_3}{\partial\xi^1}-t_5\frac{\partial w_3}{\partial\xi^2} \\ &+w_5u_1-2w_4u_2+2w_4u_3-4w_4u_6+3w_3u_9 \\ &-\tfrac{2}{\lambda_2}\gamma_7v_1+\big[-\tfrac{3}{\lambda_3}\gamma_5+(\tfrac{1}{\lambda_3})^2\overline{D}(\lambda_3)\big]v_2 \\ &+\big[-\tfrac{2}{\lambda_4}\gamma_2-\tfrac{4}{\lambda_3}\gamma_6-\tfrac{1}{\lambda_3}\gamma_8+\tfrac{1}{\lambda_3}\overline{\gamma}_8+(\tfrac{1}{\lambda_3})^2N(\lambda_3)+(\tfrac{1}{\lambda_4})^2L(\lambda_4)\big]v_3 \end{aligned}$$

Proof. Follows from the Appendices B, C, D. See, Proposition D.4. □

Remark 2.7. Write $\mathbf{A}(\Phi) = \mathbf{A}^\mu \frac{\partial}{\partial x^\mu} = ({\mathbf{A}_1}^\mu \oplus {\mathbf{A}_2}^\mu \oplus {\mathbf{A}_3}^\mu)\frac{\partial}{\partial x^\mu}$. Explicitly,

$$\mathbf{A}^1 = e_4\,\mathrm{diag}(0,0,1,0,0) \oplus e_4\,\mathrm{diag}(0,0,0,0,1,1,1,0) \oplus \begin{pmatrix} \lambda_1 e_4 & \lambda_1 e_1 & 0 & 0 & 0 \\ \lambda_1 \overline{e}_1 & \lambda_2 e_4 & \lambda_2 e_1 & 0 & 0 \\ 0 & \lambda_2 \overline{e}_1 & \lambda_3 e_4 & \lambda_3 e_1 & 0 \\ 0 & 0 & \lambda_3 \overline{e}_1 & \lambda_4 e_4 & \lambda_4 e_1 \\ 0 & 0 & 0 & \lambda_4 \overline{e}_1 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = e_3\,\mathrm{diag}(1,1,0,1,1) \oplus e_3\,\mathrm{diag}(1,1,1,1,0,0,0,1) \oplus e_3\,\mathrm{diag}(0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$
$$\mathbf{A}^4 = \quad \mathrm{diag}(0,0,1,0,0) \oplus \quad \mathrm{diag}(0,0,0,0,1,1,1,0) \oplus \quad \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,0)$$

The matrices $\mathbf{A}^\mu$ are Hermitian matrices whose entries are linear (over $\mathbb{R}$) functions of $\Phi_1 = e$. The matrix $\mathbf{A}^3 + \mathbf{A}^4$ is strictly positive definite by the requirement $e_3 > 0$ of condition $(\star)$. We see that, **(SHS)** is truly a quasilinear, symmetric hyperbolic system. An entirely similar discussion applies to $\widehat{\textbf{(SHS)}}$. See [John], [Tay], for linear and quasilinear symmetric hyperbolic systems (in the sense of Friedrichs).

Remark 2.8. By definition, the fields Φ and $\Phi^\sharp$ take values in $\mathcal{R} \cong \mathbb{R}^{31}$ and $\widehat{\mathcal{R}} \cong \mathbb{R}^{32}$, respectively. The left and right hand sides of **(SHS)** are in $\mathcal{R}$, because $\mathbf{f}_{31}$, $\mathbf{f}_{41}$, $\mathbf{f}_{51}$, $\mathbf{f}_{22}$, $\mathbf{f}_{62}$, marked by $\star$, are real. The left and right hand sides of $\widehat{\textbf{(SHS)}}$ are in $\widehat{\mathcal{R}}$. In other words, **(SHS)** is equivalent to a *real* quasilinear symmetric hyperbolic system for an $\mathbb{R}^{31}$ valued field, and $\widehat{\textbf{(SHS)}}$ is equivalent to a *real* linear homogeneous symmetric hyperbolic system for an $\mathbb{R}^{32}$ valued field.

Remark 2.9. Let $\mathfrak{P}$ be the parity transformation $(\mathfrak{P}\cdot\Phi)(x) = (-1)^A\,\Phi(x)$ with

$$A = \mathrm{diag}(1,1,0,0,0) \oplus \mathrm{diag}(0,0,1,1,1,0,0,0) \oplus \mathrm{diag}(0,1,0,1,0).$$

The field $\mathfrak{P}\cdot\Phi$ solves **(SHS)** if and only if Φ solves **(SHS)** . The constraint $(\mathfrak{P}\cdot\Phi)^\sharp = 0$ if and only if $\Phi^\sharp = 0$. Clearly, $\mathfrak{P}\circ\mathfrak{P} =$ Identity, and Φ splits naturally into 11 $\mathfrak{P}$-even and 7 $\mathfrak{P}$-odd components. If the $\mathfrak{P}$-odd components of Φ vanish at $x \in \mathcal{U}$, that is $(\mathfrak{P}\cdot\Phi)(x) = \Phi(x)$, then **(SHS)** implies that $L(e_4) = L(e_5) = 0$ at x.

Proposition 2.4. *Suppose $(\star)$. Set all the $\mathfrak{P}$-odd components, e_1, e_2, γ_3, γ_4, γ_5, w_2, w_4, and the two $\mathfrak{P}$-even components e_4, e_5 of the field Φ equal to zero, and introduce the field $\widetilde{\Phi} = e_3 \oplus (\gamma_1,\gamma_2,\gamma_6,\gamma_7,\gamma_8) \oplus (w_1,w_3,w_5)$. In this case, the frame collapses to $D = 0$, $N = \frac{\partial}{\partial u}$ and $L = e_3\frac{\partial}{\partial \underline{u}}$ and the system* **(SHS)** *reduces to*

$$\begin{aligned} \widetilde{\mathbf{A}}\big(\widetilde{\Phi}\big)\widetilde{\Phi} &= \widetilde{\mathbf{f}}\big(\widetilde{\Phi}\big) \qquad \textbf{(subSHS)} \\ \widetilde{\mathbf{A}}\big(\widetilde{\Phi}\big) &= N \oplus \mathrm{diag}(L,L,N,N,L) \oplus \mathrm{diag}(\lambda_1 N,\ \lambda_2 L + \lambda_3 N,\ \lambda_4 L) \\ \widetilde{\mathbf{f}}\big(\widetilde{\Phi}\big) &= \mathbf{f}_{31} \oplus (\mathbf{f}_{12},\mathbf{f}_{22},\mathbf{f}_{62},\mathbf{f}_{72},\mathbf{f}_{82}) \oplus (\mathbf{f}_{13},\mathbf{f}_{33},\mathbf{f}_{53}) \qquad \textit{(see, Definition 2.3)} \end{aligned}$$

It is, separately for each ξ, a quasilinear symmetric hyperbolic system for $\widetilde{\Phi}$ in the $(\underline{u},u)$ plane. The components t, u_1, u_7, u_8, u_9, v_1, v_3 of $\Phi^\sharp$ vanish. $\widehat{\textbf{(SHS)}}$ *reduces to a linear (over $\mathbb{R}$), homogeneous symmetric hyperbolic system for $\widetilde{\Phi}^{\,\sharp} = (u_2,u_3,u_4,u_5,u_6)\oplus v_2$.*

Corollary 2.1. *Suppose, Φ is a solution to any well posed problem for* **(SHS)** *, such that all its $\mathfrak{P}$-odd components and e_4, e_5 vanish initially. Then, they vanish everywhere.*

3. Symmetries

A **field transformation** S with respect to the open subsets $\mathcal{U}, \mathcal{U}'$ of $\mathbb{R}^4$ consists of

- a diffeomorphism from $\mathcal{U}$ to $\mathcal{U}'$,
- a map from fields $\Phi = (e, \gamma, w) : \mathcal{U} \to \mathcal{R}$ to fields $\Phi' = (e', \gamma', w') : \mathcal{U}' \to \mathcal{R}$,
- a map from strictly positive weight functions $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ on $\mathcal{U}$ to strictly positive weight functions $\Lambda' = (\lambda_1', \lambda_2', \lambda_3', \lambda_4')$ on $\mathcal{U}'$.

Let x, x' be Cartesian coordinates on $\mathcal{U}$ and $\mathcal{U}'$. We write

$$x' = S \cdot x \qquad \Phi'(x') = (S \cdot \Phi)(x') \qquad \Lambda'(x') = (S \cdot \Lambda)(x')$$

In this section, S will always be linear over real valued functions in its action on Φ and Λ. That is,

$$\big(S \cdot (f\Phi)\big)(x') = f(S^{-1} \cdot x')\,(S \cdot \Phi)(x') \;\;,\;\; \big(S \cdot (f\Lambda)\big)(x') = f(S^{-1} \cdot x')\,(S \cdot \Lambda)(x')$$

for all $f \in C(\mathcal{U}, \mathbb{R})$. Therefore, S acts pointwise. For this reason, it suffices to make a local analysis. For the rest of this section, we make the assumption that $x' = S \cdot x$ is a local diffeomorphism on $\mathbb{R}^4$. With this understanding, it is unnecessary to specify the domains $\mathcal{U}$ and $\mathcal{U}'$.

Definition 3.1. *A field transformation S is a* **field symmetry** *if:*

- $(\star)$ *and* $(\star\star)$ *are preserved (see, Definition 2.1).*
- *Φ satisfies* **(SHS)** *on $\mathcal{U}$ if and only if $S \cdot \Phi$ satisfies* **(SHS)** *on $\mathcal{U}'$.*
- *$\Phi^\sharp$ vanishes on $\mathcal{U}$ if and only $(S \cdot \Phi)^\sharp$ vanishes on $\mathcal{U}'$.*

It is implicit in the last two statements that the weights Λ appear on $\mathcal{U}$ and the weights $S \cdot \Lambda$ appear on $\mathcal{U}'$. For a field symmetry S, it follows that Φ is a vacuum field on $\mathcal{U}$ if and only if $S \cdot \Phi$ is a vacuum field on $\mathcal{U}'$.

As in Section 2, let $x = (x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$. We now define a number of field transformations.

Definition 3.2. Angular coordinate transformation $\mathfrak{C}$. *Let $\mathfrak{C}^1$, $\mathfrak{C}^2$ be real functions.*

$$\begin{aligned}
x' &= \mathfrak{C} \cdot x = \big(\mathfrak{C}^1(x^1, x^2),\; \mathfrak{C}^2(x^1, x^2),\; x^3,\; x^4\big) \\
e'_A(x') &= \sum_{B=1}^{2} \tfrac{\partial \mathfrak{C}^A}{\partial x^B}(x)\; e_B(x)\,\big|_{x = \mathfrak{C}^{-1} \cdot x'} && A = 1, 2 \\
e'_{A+3}(x') &= \sum_{B=1}^{2} \tfrac{\partial \mathfrak{C}^A}{\partial x^B}(x)\; e_{B+3}(x)\,\big|_{x = \mathfrak{C}^{-1} \cdot x'} && A = 1, 2 \\
(e'_3, \gamma', w', \Lambda')(x') &= (e_3, \gamma, w, \Lambda)(\mathfrak{C}^{-1} \cdot x')
\end{aligned}$$

We will also use the notation $\mathfrak{C}(\xi) = \mathfrak{C}^1(\xi) + i\mathfrak{C}^2(\xi)$, where $\xi = \xi^1 + i\xi^2$.

Definition 3.3. $U(1)$ **transformation** $\mathfrak{Z}$**.** *Let* $\zeta = \zeta(x^1, x^2) \in U(1)$.

$$x' = \mathfrak{Z} \cdot x = x$$

$$\Phi'(x') = (\mathfrak{Z} \cdot \Phi)(x') = \zeta^A \Phi(x) \quad + \quad \begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix} \oplus \begin{pmatrix} 0\\0\\ \frac{1}{2}\, D(\zeta)\\ \frac{1}{2}\, \overline{D(\zeta)}\\0\\0\\0\\ \frac{1}{2}\, \zeta^{-1} N(\zeta) \end{pmatrix}(x) \oplus \begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix} \Bigg|_{x = \mathfrak{Z}^{-1}\cdot x'}$$

$$A = \mathrm{diag}\big(1,1,0,0,0,\;\; 2,0,1,-1,-1,0,-2,0,\;\; 2,1,0,-1,-2\big)$$

$$\Lambda'(x') = (\mathfrak{Z} \cdot \Lambda)(x') = \Lambda(x)\,\big|_{x = \mathfrak{Z}^{-1}\cdot x'}$$

Here $D(\zeta) = (e_1 \frac{\partial}{\partial x^1} + e_2 \frac{\partial}{\partial x^2})(\zeta)$ *and* $N(\zeta) = (e_4 \frac{\partial}{\partial x^1} + e_5 \frac{\partial}{\partial x^2})(\zeta)$.

Definition 3.4. Global Isotropic Scaling $\mathfrak{J}$**.** *Let* $\mathfrak{J} > 0$ *be a constant.*

$$\begin{aligned} x' &= \mathfrak{J} \cdot x = (x^1,\, x^2,\, \mathfrak{J}x^3,\, \mathfrak{J}x^4) \\ \Phi'(x') &= (\mathfrak{J} \cdot \Phi)(x') \;=\; \mathfrak{J}^A \Phi(x)\,\big|_{x = \mathfrak{J}^{-1}\cdot x'} \\ A &= \;(-1)\,\mathrm{diag}\big(1,1,0,1,1,\;\; 1,1,1,1,1,1,1,1,\;\; 2,2,2,2,2\big) \\ \Lambda'(x') &= (\mathfrak{J} \cdot \Lambda)(x') = \Lambda(x)\,\big|_{x = \mathfrak{J}^{-1}\cdot x'} \end{aligned}$$

Definition 3.5. Global Anisotropic Scaling $\mathfrak{A}$**.** *Let* $\mathfrak{A} \neq 0$.

$$\begin{aligned} x' &= \mathfrak{A} \cdot x = \big(\tfrac{1}{\mathfrak{A}}x^1,\, \tfrac{1}{\mathfrak{A}}x^2,\, x^3,\, \mathfrak{A}^2 x^4\big) \\ \Phi'(x') &= (\mathfrak{A} \cdot \Phi)(x') \;=\; \mathfrak{A}^A \Phi(x)\big|_{x = \mathfrak{A}^{-1}\cdot x'} \\ A &= \;(-1)\,\mathrm{diag}\big(2,2,0,3,3,\;\; 0,0,1,1,1,2,2,2,\;\; 0,1,2,3,4\big) \\ \Lambda'(x') &= (\mathfrak{A} \cdot \Lambda)(x') = \mathrm{diag}(1, \mathfrak{A}^2, \mathfrak{A}^4, \mathfrak{A}^6)\, \Lambda(x)\,\big|_{x = \mathfrak{A}^{-1}\cdot x'} \end{aligned}$$

The transformation $\mathfrak{A}$ plays a central role in this paper. For a sample calculation, see the proof of Proposition 4.1.

Proposition 3.1. $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ *are field symmetries.*

Proposition 3.1 is proven in Appendix E. We will now define an additional transformation. It is a composition of two field symmetries, and therefore itself a field symmetry.

Definition 3.6. Pole-Flip transformation Flip$_\alpha$. *Let* $\alpha \neq 0$ *be a constant. The Pole-Flip transformation* **Flip**$_\alpha$ *is the composition of a* $U(1)$ *transformation and an angular coordinate transformation. Precisely,*

$$\mathbf{Flip}_\alpha \;=\; \mathfrak{Z} \circ \mathfrak{C} \qquad \textit{where} \qquad \zeta(\xi) = -\frac{\xi}{\bar{\xi}} \qquad \mathfrak{C}(\xi) = \frac{\alpha^2}{\xi}$$

and $\xi = \xi^1 + i\xi^2$ *and* $\mathfrak{C}(\xi) = \mathfrak{C}^1(\xi) + i\mathfrak{C}^2(\xi)$.

Remark 3.1. With this choice of ζ and $\mathfrak{C}$, we have $\mathfrak{C}^{-1} \circ \mathfrak{Z} \circ \mathfrak{C} = \mathfrak{Z}^{-1}$ and $\mathfrak{C} \circ \mathfrak{C} =$ Identity. Therefore, $\mathbf{Flip}_\alpha \circ \mathbf{Flip}_\alpha = \mathrm{Identity}$. The field symmetry $\mathbf{Flip}_\alpha$ will be used to match constructions between two "angular" coordinate patches. For Minkowski, ξ will be stereographic coordinates (scaled by α) based on the north and south poles.

4. The Doubly Scaled Minkowski Field $\mathcal{M}_{a,\mathfrak{A}}$

Fix the coordinates $(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$ as in the preceding sections. For all pairs $\mu, \lambda > 0$, set

$$\begin{aligned} \mathbf{Strip}(\mu,\lambda) &= \mathbb{R}^2 \times (0,\mu) \times (-\infty, -\lambda^{-1}) \\ \mathbf{Strip}_\infty = \mathbf{Strip}(\infty,\infty) &= \mathbb{R}^2 \times (0,\infty) \times (-\infty, 0) \end{aligned} \tag{4.1}$$

Definition 4.1. *For all $a, \mathfrak{A} \neq 0$, let $\mathcal{M}_{a,\mathfrak{A}} : \mathbf{Strip}_\infty \to \mathcal{R}$ (see (2.1)) be the field*

$$\mathcal{M}_{a,\mathfrak{A}} = \begin{pmatrix} \rho_{a,\mathfrak{A}}^{-1}\, \mathbf{e}_{a,\mathfrak{A}} \\ i\,\rho_{a,\mathfrak{A}}^{-1}\, \mathbf{e}_{a,\mathfrak{A}} \\ 1 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \rho_{a,\mathfrak{A}}^{-1}\, \mathfrak{A}^2 \\ \rho_{a,\mathfrak{A}}^{-1}\, \boldsymbol{\lambda}_{a,\mathfrak{A}} \\ \rho_{a,\mathfrak{A}}^{-1}\, \overline{\boldsymbol{\lambda}}_{a,\mathfrak{A}} \\ 0 \\ -\rho_{a,\mathfrak{A}}^{-1} \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

Here $\xi = \xi^1 + i\xi^2$ and

$$\rho_{a,\mathfrak{A}}(u,\underline{u}) = \mathfrak{A}^2\underline{u} - u \qquad \mathbf{e}_{a,\mathfrak{A}}(\xi) = \tfrac{a}{2}\big(1 + \tfrac{\mathfrak{A}^2}{a^2}|\xi|^2\big) \qquad \boldsymbol{\lambda}_{a,\mathfrak{A}}(\xi) = -\tfrac{\mathfrak{A}^2}{2a}\,\xi \tag{4.3}$$

We will often consciously suppress the subscripts a, $\mathfrak{A}$ on the functions ρ, $\mathbf{e}$ and $\boldsymbol{\lambda}$. Set

$$S(u,\underline{u}) = \tfrac{u^2}{\rho} + u. \tag{4.4}$$

The decomposition $\frac{1}{\rho} = -\frac{1}{u} + \frac{S}{u^2}$ will be used over and over again.

Proposition 4.1. *For all $a, \mathfrak{A} \neq 0$:*

- *(a) $\mathcal{M}_{a,\mathfrak{A}} = (\mathfrak{C} \circ \mathfrak{A}) \cdot \mathcal{M}_{1,1}$ on $\mathbf{Strip}_\infty$, where $\mathfrak{C}(\xi) = a\,\xi$.*
- *(b) $\mathcal{M}_{a,\mathfrak{A}} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{M}_{a,\mathfrak{A}}$ on $\mathbf{Strip}_\infty \cap \{\xi \neq 0\}$.*
- *(c) $\mathcal{M}_{a,\mathfrak{A}} = \mathfrak{J} \cdot \mathcal{M}_{a,\mathfrak{A}}$ on $\mathbf{Strip}_\infty$, for all $\mathfrak{J} > 0$.*
- *(d) $\mathcal{M}_{a,\mathfrak{A}}$ is a* vacuum field *on $\mathbf{Strip}_\infty$ (see, Definition 2.2).*
- *(e) The Lorentzian manifold associated to $\mathcal{M}_{a,\mathfrak{A}}$ is isometric to the open subset of Minkowski space given by* (4.5) *below. For this reason, we refer to $\mathcal{M}_{a,\mathfrak{A}}$ as the* **doubly scaled Minkowski field**.

Here, $\mathfrak{C}$, $\mathfrak{A}$, $\mathfrak{J}$ and $\mathbf{Flip}_\alpha$ are field transformations defined in Section 3.

Proof. For (a), set $C_a = \operatorname{diag}(a,a,1,a,a) \oplus \mathbb{1}_8 \oplus \mathbb{1}_5$ and let A be the matrix in Definition 3.5. Then,

$$\begin{aligned}
\big((\mathfrak{C}\circ\mathfrak{A})\cdot\mathcal{M}_{1,1}\big)(x'') &= \big(\mathfrak{C}\cdot(\mathfrak{A}\cdot\mathcal{M}_{1,1})\big)(x'')\\
&= C_a\,(\mathfrak{A}\cdot\mathcal{M}_{1,1})(x')\,\big|_{x'=\mathfrak{C}^{-1}\cdot x''}\\
&= C_a\,\mathfrak{A}^A\,\mathcal{M}_{1,1}(x)\big|_{x=\mathfrak{A}^{-1}\cdot x'=\mathfrak{A}^{-1}\cdot(\mathfrak{C}^{-1}\cdot x'')}\\
&= C_a\,\mathfrak{A}^A\,\mathcal{M}_{1,1}\big(\tfrac{\mathfrak{A}}{a}\xi'', \underline{u}'', \tfrac{1}{\mathfrak{A}^2}u''\big)\\
&= \mathcal{M}_{a,\mathfrak{A}}(x'')
\end{aligned}$$

Similarly for (b) and (c). Part (d) is also verified by direct calculation. It suffices to check (d) for $\mathcal{M}_{1,1}$, because the general case follows from (a) and Proposition 3.1. Recall that the definition of a vacuum field is independent of the choice of weight functions $\lambda_1,\ldots,\lambda_4$. For (e), see Remark 4.1 below. $\square$

Remark 4.1. The Riemann curvature tensor of the Lorentzian manifold associated to the vacuum field $\mathcal{M}_{a,\mathfrak{A}} = (e,\gamma,w)$ on **Strip**$_\infty$ vanishes, because $w=0$ (see, Definition 2.2). It is isometric to the open subset of Minkowski space given by

$$\Big\{(X^0,\mathbf{X})\in\mathbb{R}\times\mathbb{R}^3 \;\Big|\; |X^0|<|\mathbf{X}|,\ \mathbf{X}\notin\{0\}\times\{0\}\times[0,\infty)\Big\} \tag{4.5}$$

where $(X^0,\mathbf{X})$ are the standard Minkowski coordinates, and

$$X^0 = \frac{1}{\sqrt{2}\,|\mathfrak{A}|}(\mathfrak{A}^2\underline{u}+u) \qquad \begin{pmatrix}X^1\\X^2\\X^3\end{pmatrix} = \frac{1}{\sqrt{2}\,|\mathfrak{A}|}\,\frac{\mathfrak{A}^2\underline{u}-u}{1+\frac{\mathfrak{A}^2}{a^2}|\xi|^2}\begin{pmatrix}\frac{2\mathfrak{A}}{a}\xi^1\\ \frac{2\mathfrak{A}}{a}\xi^2\\ -1+\frac{\mathfrak{A}^2}{a^2}|\xi|^2\end{pmatrix}$$

The level sets of $u = 2^{-\frac{1}{2}}|\mathfrak{A}|\,\big(X^0-|\mathbf{X}|\big)<0$ and $\underline{u} = 2^{-\frac{1}{2}}|\frac{1}{\mathfrak{A}}|\big(X^0+|\mathbf{X}|\big)>0$ are null hypersurfaces. They intersect in a standard sphere of radius $|\mathbf{X}| = 2^{-\frac{1}{2}}|\frac{1}{\mathfrak{A}}|\rho$, with the north pole removed, on which $\frac{\mathfrak{A}}{a}\,\xi$ is the standard stereographic coordinate system. The southern hemisphere corresponds to $|\xi|<|\frac{a}{\mathfrak{A}}|$.

Remark 4.2. The limit $\mathcal{M}_{0,0} = \lim_{\mathfrak{A}\downarrow 0}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ exists on **Strip**$_\infty$. By taking the limit of (d) in Proposition 4.1, it is a solution to **(SHS)** with $\mathcal{M}^\sharp_{0,0} = 0$. Observe that the associated frame is degenerate, because $D=0$ here. (See, Proposition 2.4.)

Remark 4.3. For each $a\neq 0$, the limit $\mathcal{M}_{a,0} = \lim_{\mathfrak{A}\downarrow 0}\mathcal{M}_{a,\mathfrak{A}}$ exists on **Strip**$_\infty$. By taking the limit of (d) in Proposition 4.1, it is a vacuum field. The Lorentzian manifold associated to $\mathcal{M}_{a,0}$ is isometric to the open subset of Minkowski space given by $\{|X^0|<|\mathbf{X}|,\ X^0+X^3<0\}$:

$$\tfrac{1}{\sqrt{2}}(X^0+X^3) = u \qquad \tfrac{1}{\sqrt{2}}(X^0-X^3) = \underline{u}+u\,\tfrac{1}{a^2}\,|\xi|^2 \qquad \begin{pmatrix}X^1\\X^2\end{pmatrix} = -\sqrt{2}\,u\begin{pmatrix}\frac{1}{a}\xi^1\\ \frac{1}{a}\xi^2\end{pmatrix}$$

Observe that $-(X^0)^2+|\mathbf{X}|^2 = -2u\underline{u}$. The field $\mathcal{M}_{a,0}$ is independent of ξ and $\underline{u}$, that is, translation invariant in these directions. The level sets of u and $\underline{u}$ are null hypersurfaces. They intersect in standard Euclidean planes.

5. The Far (Weak) Field Ansatz

We now formulate a physically interesting initial value problem, motivated by [Chr], for **(SHS)** (see, Section 2). It is set up so that $\Phi^\sharp$ always vanishes, that is, Φ is a vacuum field. Under appropriate conditions, the corresponding vacuum spacetime (see, Remark 2.3) contains trapped spheres.

Fix the coordinate system $(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$ on the infinitely wide strip $\mathbf{Strip}_\infty \subset \mathbb{R}^4$, see (4.1), and make the far ("past null infinity") field Ansatz

$$\Phi(x) \;=\; \mathcal{M}_{a,\mathfrak{A}}(x) \;+\; u^{-M}\Psi(x) \tag{5.1}$$

on $\mathcal{U}$. See, (4.2) for the definition of $\mathcal{M}_{a,\mathfrak{A}}(x)$. Here,

$$\begin{aligned} M \;&=\; \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(1,2,2,2,2,2,2,3) \oplus \mathrm{diag}(1,2,3,4,4) \\ \Psi(x) \;&=\; \big(\Psi_1(x),\, \Psi_2(x),\, \Psi_3(x)\big) \;=\; \big(f(x),\, \omega(x),\, z(x)\big) \;\in\; \mathcal{R} \end{aligned}$$

Our basic Ansatz (5.1), Minkowski plus asymptotically small corrections (assuming, $\Psi = \mathcal{O}(1)$ as $u \to -\infty$), is completely naive. The only subtlety, lies in the choice of the diagonal matrix M that prescribes the far field asymptotics of the system, and is ultimately a statement about the physics of propagating gravitational waves. Anyone who has made formal or rigorous perturbative calculations or constructions in classical or quantum physics knows from experience that one must "play with the expansion" until, "one sees what is going on". We have followed this traditional route to the matrix M. However, the only real justification is that it works.

Definition 5.1. *Let* $\mathbf{DATA}(\xi, \underline{u}) : \mathbb{R}^2 \times (0,\infty) \to \mathbb{C}$ *be smooth and vanish when* $\underline{u} < \underline{u}_0$, *for some fixed* $\underline{u}_0 > 0$. *Set, step by step down the left column and then down the right column,* $\Psi(0)(\xi,\underline{u}) = \big(f(0)(\xi,\underline{u}), \omega(0)(\xi,\underline{u}), z(0)(\xi,\underline{u})\big) \in \mathcal{R}$ *equal to*

$$\begin{aligned}
\omega_1(0) &= \mathbf{DATA} \\
\omega_7(0) &= -\partial_{\underline{u}}^{-1}\overline{\omega_1(0)} \\
z_1(0) &= -\tfrac{\partial}{\partial \underline{u}}\omega_1(0) \\
z_2(0) &= 2\big(\mathbf{e}\tfrac{\partial}{\partial\xi} + 2\overline{\boldsymbol{\lambda}}\big)\,\partial_{\underline{u}}^{-1} z_1(0) \\
z_3(0) &= 2\big(\mathbf{e}\tfrac{\partial}{\partial\xi} + \overline{\boldsymbol{\lambda}}\big)\,\partial_{\underline{u}}^{-1} z_2(0) - \partial_{\underline{u}}^{-1}\big(\omega_7(0) z_1(0)\big) \\
z_4(0) &= 2\,\mathbf{e}\tfrac{\partial}{\partial\xi}\,\partial_{\underline{u}}^{-1} z_3(0) - 2\partial_{\underline{u}}^{-1}\big(\omega_7(0) z_2(0)\big) \\
\omega_3(0) &= -\partial_{\underline{u}}^{-1} z_2(0) - \overline{\boldsymbol{\lambda}}\,\partial_{\underline{u}}^{-1}\omega_1(0) \\
\omega_5(0) &= -\partial_{\underline{u}}^{-1}\overline{z_2(0)} \\
\omega_8(0) &= \partial_{\underline{u}}^{-1} z_3(0) - 4i\,\partial_{\underline{u}}^{-1}\Im\big(\boldsymbol{\lambda}\,\omega_5(0)\big)
\end{aligned}
\qquad
\begin{aligned}
z_5(0) &= 0 \\
\omega_2(0) &= -\partial_{\underline{u}}^{-1}|\omega_1(0)|^2 \\
\omega_4(0) &= -\boldsymbol{\lambda}\,\partial_{\underline{u}}^{-1}\overline{\omega_1(0)} \\
\omega_6(0) &= 0 \\
f_1(0) &= \,\mathbf{e}\,\partial_{\underline{u}}^{-1}\omega_1(0) \\
f_2(0) &= -i\,\mathbf{e}\,\partial_{\underline{u}}^{-1}\omega_1(0) \\
f_3(0) &= -\Re\,\omega_8(0) \\
f_4(0) &= -4\,\mathbf{e}\,\partial_{\underline{u}}^{-1}\,\Re\,\omega_5(0) \\
f_5(0) &= 4\,\mathbf{e}\,\partial_{\underline{u}}^{-1}\,\Im\,\omega_5(0)
\end{aligned}$$

where $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$ *and* $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$ *are defined in* (4.3) *and* $\frac{\partial}{\partial\xi} = \frac{1}{2}\big(\frac{\partial}{\partial\xi^1} - i\frac{\partial}{\partial\xi^2}\big)$, *and*

$$\big(\partial_{\underline{u}}^{-1} g\big)(\underline{u}) = \int_0^{\underline{u}} \mathrm{d}\underline{u}'\, g\big(\underline{u}'\big). \tag{5.2}$$

(Asymptotic) Characteristic Initial Conditions: Informally, the initial conditions for the field Φ on $\mathbf{Strip}_\infty$ are:

$$\lim_{u\to-\infty} \Psi(\xi,\underline{u},u) \;=\; \Psi(0)(\xi,\underline{u}) \tag{5.3a}$$

$$\Psi(\xi,\underline{u},u) \;=\; 0 \qquad \text{when } \underline{u} < \underline{u}_0 \tag{5.3b}$$

Remark 5.1. The function **DATA** is arbitrary, there are no hidden symmetry assumptions. For the construction of classical solutions, we will assume a transparent smallness condition (see, Theorem 8.1).

Definition 5.2. *Let* $a,\mathfrak{A} \neq 0$. *Let*

$$[\,\mathcal{M}_{a,\mathfrak{A}}\,](x) \;=\; \textstyle\sum_{k=0}^{\infty} (\frac{1}{u})^k\, \mathcal{M}_{a,\mathfrak{A}}(k)(\xi,\underline{u})$$

be the formal expansion in $\frac{1}{u}$ *for the Minkowski vacuum field* $\mathcal{M}_{a,\mathfrak{A}}$, *see* (4.2), *in which*

$$[\,\tfrac{1}{\rho}\,] \;=\; -\tfrac{1}{u} + \tfrac{1}{u^2}\,[\,S\,] \qquad\qquad [\,S\,] \;=\; -\textstyle\sum_{k=0}^{\infty} (\frac{1}{u})^k\, \mathfrak{A}^{2(k+1)}\, \underline{u}^{k+1}. \tag{5.4}$$

In Section 6 we construct a unique formal series $[\,\Psi\,](x) = \sum_{k=0}^{\infty} (\frac{1}{u})^k\, \Psi(k)\big(\xi,\underline{u}\big)$ on $\mathbf{Strip}_\infty \subset \mathbb{R}^4$ such that $[\,\Phi\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,] + u^{-M}[\,\Psi\,]$ is a formal power series solution to **(SHS)** satisfying (5.3a) and (5.3b). In particular, $\Psi(k)(\xi,\underline{u}) = 0$ when $\underline{u} < \underline{u}_0$ for all $k \geq 0$. By construction, the associated formal constraint field $[\,\Phi^\sharp\,]$ vanishes "as $u \to -\infty$", and therefore identically because it is a formal solution to $\widehat{\textbf{(SHS)}}$.

Remark 5.2. Equation (5.3b) stipulates that Φ coincides with the Minkowski vacuum field $\mathcal{M}_{a,\mathfrak{A}}$ when $\underline{u} < \underline{u}_0$. On the other hand, (5.3a) is an asymptotic initial condition at "past null infinity" $u \to -\infty$. Where does the prescription for $\Psi(0)(\xi,\underline{u})$ in Definition 5.1 come from? The system **(SHS)** (see, Definition 2.3) for $[\,\Phi\,]$ is equivalent to a recursive hierarchy of inhomogeneous linear equations for $\Psi(k)$, $k \geq 1$, and equations for the initial term $\Psi(0)$. Additional equations for $\Psi(0)$ have to be satisfied so that the formal constraint field $[\,\Phi^\sharp\,]$ (see, Definition 2.4) vanishes "as $u \to -\infty$". Altogether, there are just enough equations to completely determine all but one of the components of $\Psi(0)$, namely $\omega_1(0) = $ **DATA**. Our prescription amounts to an "upper triangular" arrangement of these equations that generate $\Psi(0)$ out of **DATA**. See, Section 6.

The goal for the rest of this section is to bring the equations of Section 2 into a relevant / irrelevant form that exhibits the essential constituents that have to be treated carefully, and sweeps everything else into "generic terms" that we don't need to know much about.

Proposition 5.1. *In this proposition, ignore Definition 2.4, and regard* $\Phi(x)$ *and* $\Phi^\sharp(x)$ *as independent, sufficiently differentiable fields on* $\mathbf{Strip}_\infty$ *with values in* $\mathcal{R}$ *and* $\widehat{\mathcal{R}}$, *respectively. Set*

$$M = \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(1,2,2,2,2,2,2,3) \oplus \mathrm{diag}(1,2,3,4,4) \tag{5.5a}$$

$$E = \mathrm{diag}(4,4,4,6,6) \oplus \mathrm{diag}(2,4,4,4,4,4,4,6) \oplus \mathrm{diag}(0,0,0,0,0) \tag{5.5b}$$

$$M^\sharp = \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(2,2,2,2,2,2,3,3,3) \oplus \mathrm{diag}(0,-1,-2) \tag{5.5c}$$

$$E^\sharp = \mathrm{diag}(4,4,4,6,6) \oplus \mathrm{diag}(4,4,4,4,4,4,6,6,6) \oplus \mathrm{diag}(2,2,2) \tag{5.5d}$$

and

$$\Phi(x) = \mathcal{M}_{a,\mathfrak{A}}(x) + u^{-M}\,\Psi(x) \qquad \Psi(x) \in \mathcal{R} \tag{5.6a}$$

$$\Phi^\sharp(x) = u^{-M^\sharp}\,\Psi^\sharp(x) \qquad \Psi^\sharp(x) \in \widehat{\mathcal{R}} \tag{5.6b}$$

$$\lambda_j(x) = u^{2j} \qquad j = 1,2,3,4 \quad \textit{(see, Definition 2.3)} \tag{5.6c}$$

The systems (see, Section 2) $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$ *and* $\widehat{\mathbf{A}}(\Phi)\,\Phi^\sharp = \widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\,\Phi^\sharp$ *for* Φ *and* $\Phi^\sharp$ *are equivalent to the following systems for* Ψ *and* $\Psi^\sharp$*:*

$$\mathbf{A}_{a,\mathfrak{A}}(x,\Psi)\,\Psi \;=\; \mathbf{f}_{a,\mathfrak{A}}(x,\Psi) \tag{5.7a}$$

$$\widehat{\mathbf{A}}_{a,\mathfrak{A}}(x,\Psi)\,\Psi^\sharp \;=\; \widehat{\mathbf{f}}_{a,\mathfrak{A}}(x,\Psi,\partial_x\Psi)\,\Psi^\sharp \tag{5.7b}$$

where $\mathbf{A}_{a,\mathfrak{A}}(x,\Psi) = \mathbf{A}^\mu_{a,\mathfrak{A}}(x,\Psi)\frac{\partial}{\partial x^\mu}$ *and* $\widehat{\mathbf{A}}_{a,\mathfrak{A}}(x,\Psi) = \widehat{\mathbf{A}}^\mu_{a,\mathfrak{A}}(x,\Psi)\frac{\partial}{\partial x^\mu}$ *and*

$$\mathbf{A}^\mu_{a,\mathfrak{A}}(x,\Psi) = u^E\big(u^{-M}\mathbf{A}^\mu(\Phi)u^{-M}\big) \tag{5.8a}$$

$$\mathbf{f}_{a,\mathfrak{A}}(x,\Psi) = u^{E-M}\Big(-\mathbf{A}^\mu(\Phi)\big(\tfrac{\partial}{\partial x^\mu}u^{-M}\big)\,\Psi + \mathbf{f}(\Phi) - \mathbf{A}^\mu(\Phi)\tfrac{\partial}{\partial x^\mu}\mathcal{M}_{a,\mathfrak{A}}\Big) \tag{5.8b}$$

$$\widehat{\mathbf{A}}^\mu_{a,\mathfrak{A}}(x,\Psi) = u^{E^\sharp}\big(u^{-M^\sharp}\,\widehat{\mathbf{A}}^\mu(\Phi)\,u^{-M^\sharp}\big) \tag{5.8c}$$

$$\widehat{\mathbf{f}}_{a,\mathfrak{A}}(x,\Psi,\partial_x\Psi) = u^{E^\sharp - M^\sharp}\Big(-\widehat{\mathbf{A}}^\mu(\Phi)\,\big(\tfrac{\partial}{\partial x^\mu}u^{-M^\sharp}\big) + \widehat{\mathbf{f}}(\Phi,\partial_x\Phi)\,u^{-M^\sharp}\Big) \tag{5.8d}$$

In (5.8)*,* Φ*,* $\partial_x\Phi$ *have to be expressed in terms of* Ψ*,* $\partial_x\Psi$ *using* (5.6a)*. We will sometimes drop the* a*,* $\mathfrak{A}$ *and write* $\mathbf{A}(x,\Psi)$*,* $\mathbf{f}(x,\Psi)$*,* $\widehat{\mathbf{A}}(x,\Psi)$*,* $\widehat{\mathbf{f}}(x,\Psi,\partial_x\Psi)$*. They are notationally distinguished from* $\mathbf{A}(\Phi)$*,* $\mathbf{f}(\Phi)$*,* $\widehat{\mathbf{A}}(\Phi)$*,* $\widehat{\mathbf{f}}(\Phi,\partial_x\Phi)$ *by the number of arguments.*

Remark 5.3. The matrices $\mathbf{A}^\mu(x,\Psi)$, $\widehat{\mathbf{A}}^\mu(x,\Psi)$ are Hermitian, so that (5.7a) and (5.7b) are also symmetric hyperbolic. They are affine linear (over $\mathbb{R}$) functions of the field Ψ. The linear (over $\mathbb{R}$) transformation $\widehat{\mathbf{f}}(x,\Psi,\partial_x\Psi)$ depends affine linearly (over $\mathbb{R}$) on $\Psi \oplus \partial_x\Psi$. On the other hand, $\mathbf{f}(x,\Psi)$ is a quadratic polynomial in the components of Ψ, $\overline{\Psi}$ without constant term. There is no constant term, because $\mathcal{M}_{a,\mathfrak{A}}$ is a vacuum field. By direct inspection, neither derivatives of $\mathbf{e}_{a,\mathfrak{A}}$ nor derivatives of $\boldsymbol{\lambda}_{a,\mathfrak{A}}$ appear in the term $\mathbf{A}^\mu(\Phi)\frac{\partial}{\partial x^\mu}\mathcal{M}_{a,\mathfrak{A}}$. See, (2.5) and (4.2).

Definition 5.3. *Let* S *be defined as in equation* (4.4)*.*

- $\mathcal{P}$ *is a generic symbol for a quadratic polynomial in the components of the fields* Ψ *and* $\overline{\Psi}$ *without constant term, whose coefficients are (complex) polynomials in* $\frac{1}{u}$*,* $\mathfrak{A}$*,* S*,* $\mathbf{e}_{a,\mathfrak{A}}$*,* $\boldsymbol{\lambda}_{a,\mathfrak{A}}$*,* $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$*.*
- $\mathcal{P}^\sharp$ *is a generic symbol for a polynomial in the components of the fields* Ψ *and* $\overline{\Psi}$ *and all their first order coordinate derivatives, whose coefficients are (complex) polynomials in* $\frac{1}{u}$*,* $\mathfrak{A}$*,* S*,* $\mathbf{e}_{a,\mathfrak{A}}$*,* $\boldsymbol{\lambda}_{a,\mathfrak{A}}$*,* $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$*, and all their first order coordinate derivatives.*

We use the same symbols $\mathcal{P}$ *and* $\mathcal{P}^\sharp$ *for a vector or matrix all of whose entries are polynomials of this kind.*

Remark 5.4. The vector fields D, N and L corresponding to $\Phi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi$ are:

$$\begin{aligned} D &= -\tfrac{2}{u}\,\mathbf{e}_{a,\mathfrak{A}}\,\tfrac{\partial}{\partial\overline{\xi}} + \tfrac{2}{u^2}\,\mathbf{e}_{a,\mathfrak{A}}\,S\,\tfrac{\partial}{\partial\overline{\xi}} + \tfrac{1}{u^2}\big(f_1\tfrac{\partial}{\partial\xi^1} + f_2\tfrac{\partial}{\partial\xi^2}\big) \\ N &= \tfrac{\partial}{\partial u} + \tfrac{1}{u^3}\big(f_4\tfrac{\partial}{\partial\xi^1} + f_5\tfrac{\partial}{\partial\xi^2}\big) \\ L &= \tfrac{\partial}{\partial\underline{u}} + \tfrac{1}{u^2}\,f_3\,\tfrac{\partial}{\partial\underline{u}} \end{aligned} \tag{5.9}$$

Here, $\frac{\partial}{\partial\overline{\xi}} = \frac{1}{2}(\frac{\partial}{\partial\xi^1} + i\frac{\partial}{\partial\xi^2})$.

Below, $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$.

Proposition 5.2. *Let $\Psi = (f,\omega,z)$. The system* (5.7a) *takes the relevant/irrelevant form:*

$$L\begin{pmatrix} f_1\\ f_2\\ f_4\\ f_5\\ \omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_8 \end{pmatrix} = \begin{pmatrix} \mathbf{e}\,\omega_1\\ -i\,\mathbf{e}\,\omega_1\\ 2\,\mathbf{e}\,\Re(\omega_4-\overline{\omega}_3-\omega_5)\\ -2\,\mathbf{e}\,\Im(\omega_4-\overline{\omega}_3-\omega_5)\\ -z_1\\ -|\omega_1|^2\\ -z_2-\overline{\boldsymbol{\lambda}}\,\omega_1\\ -\boldsymbol{\lambda}\,\overline{\omega}_1\\ z_3+2i\,\Im\left(\boldsymbol{\lambda}\,(\omega_4-\omega_5-\overline{\omega}_3)\right) \end{pmatrix} + \frac{1}{u}\mathcal{P} \tag{5.10a}$$

$$N\begin{pmatrix} f_3\\ \omega_5\\ \omega_6\\ \omega_7 \end{pmatrix} = \frac{1}{u}\begin{pmatrix} 2\,(f_3+\Re\,\omega_8)\\ \omega_4+\omega_5-\overline{\omega}_3\\ 0\\ 0 \end{pmatrix} + \frac{1}{u^2}\mathcal{P} \tag{5.10b}$$

$$\begin{pmatrix} N & \frac{1}{u}D & 0 & 0 & 0\\ \frac{1}{u}\overline{D} & N+\frac{1}{u^2}L & \frac{1}{u}D & 0 & 0\\ 0 & \frac{1}{u}\overline{D} & N+\frac{1}{u^2}L & \frac{1}{u}D & 0\\ 0 & 0 & \frac{1}{u}\overline{D} & N+\frac{1}{u^2}L & D\\ 0 & 0 & 0 & \overline{D} & L \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5 \end{pmatrix} = \frac{1}{u}\begin{pmatrix} 0\\ 0\\ 0\\ 4\boldsymbol{\lambda}z_5\\ \mathfrak{A}^2z_5-2\overline{\boldsymbol{\lambda}}z_4-3\omega_7z_3 \end{pmatrix} + \frac{1}{u^2}\mathcal{P} \tag{5.10c}$$

Proof. By direct (machine) calculation. □

Proposition 5.3. *Let $\Psi^\sharp = (s,p,y)$, see* (5.6b)*, and recall Definition 2.4. Then,*

$$\begin{pmatrix} s_1\\ s_2\\ s_3 \end{pmatrix} = \mathcal{P}^\sharp \qquad \begin{pmatrix} s_4\\ s_5 \end{pmatrix} = -u\,N\begin{pmatrix} \overline{f}_1\\ \overline{f}_2 \end{pmatrix} + \mathcal{P}^\sharp \tag{5.11a}$$

$$\begin{pmatrix} p_1\\ p_2\\ p_3 \end{pmatrix} = \mathcal{P}^\sharp \qquad \begin{pmatrix} p_4\\ p_7\\ p_8 \end{pmatrix} = -u\,N\begin{pmatrix} \omega_1\\ \omega_3\\ -\omega_4 \end{pmatrix} + \mathcal{P}^\sharp \tag{5.11b}$$

$$\begin{pmatrix} p_5\\ p_6\\ p_9 \end{pmatrix} = \mathcal{P}^\sharp = \begin{pmatrix} -L(\omega_7)-\overline{\omega}_1\\ L(\omega_6)\\ u\,D(\omega_7)-u\,\overline{D}(\omega_6)+\omega_4-\overline{\omega}_3-4\boldsymbol{\lambda}\,\omega_7 \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp \tag{5.11c}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathcal{P}^\sharp = \begin{pmatrix} u\,\overline{D} - 4\overline{\boldsymbol{\lambda}} & L & 0 & 0 \\ \omega_7 & u\,\overline{D} - 2\overline{\boldsymbol{\lambda}} & L & 0 \\ 0 & 2\,\omega_7 & u\,\overline{D} & L \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp \tag{5.11d}$$

Proof. By direct (machine) calculation. □

Remark 5.5. Every generic symbol $\mathcal{P}^\sharp$ that appears in (5.11), has no constant term as a polynomial in the components of Ψ, $\overline{\Psi}$ and their first coordinate derivatives. There is no constant term, because $\mathcal{M}_{a,\mathfrak{A}}$ is a vacuum field.

Proposition 5.4. *Let* $\Psi^\sharp = (s,p,y)$, *see* (5.6b). *The dual system* (5.7b) *takes the relevant / irrelevant form:*

$$L\begin{pmatrix} s_1 \\ s_2 \\ s_4 \\ s_5 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{e}\,(\overline{p}_4 - p_5) + \mathbf{e}\,(\overline{p}_6 - p_3) \\ i\,\mathbf{e}\,(\overline{p}_4 - p_5) - i\,\mathbf{e}\,(\overline{p}_6 - p_3) \\ y_1 \\ 0 \\ 0 \\ 0 \\ \boldsymbol{\lambda}(p_6 - \overline{p}_3) - \overline{\boldsymbol{\lambda}}(p_4 - \overline{p}_5) \\ -\overline{\boldsymbol{\lambda}}(\overline{p}_6 - p_3) + \boldsymbol{\lambda}(\overline{p}_4 - p_5) \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp\Psi^\sharp \tag{5.12a}$$

$$N\begin{pmatrix} s_3 \\ p_5 \\ p_6 \\ p_9 \end{pmatrix} = \frac{1}{u}\begin{pmatrix} s_3 + p_7 + \overline{p}_8 \\ \overline{p}_4 \\ \overline{p}_3 \\ \overline{p}_7 + p_8 \end{pmatrix} + \frac{1}{u^2}\mathcal{P}^\sharp\Psi^\sharp \tag{5.12b}$$

$$\begin{pmatrix} N + \frac{1}{u^2}L & \frac{1}{u}D & 0 \\ \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L & \frac{1}{u}D \\ 0 & \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{u^2}\mathcal{P}^\sharp\Psi^\sharp \tag{5.12c}$$

Above, the symbols $\mathcal{P}^\sharp$ *are linear over* $\mathbb{R}$ *generic transformations, in the sense of Definition 5.3.*

Proof. By direct (machine) calculation. □

Remark 5.6. The overall factors u^E and $u^{E^\sharp}$ appear in (5.7a) and (5.7b), so that these systems are line by line (up to a permutation of the lines) equivalent to their relevant / irrelevant counterparts in Propositions 5.2 and 5.4.

6. Formal Solutions

In this section we consider formal power series

$$[\,\Psi\,](x) = \sum_{k=0}^{\infty} (\tfrac{1}{u})^k\, \Psi(k)(\xi, \underline{u}) \tag{6.1}$$

on $\textbf{Strip}_\infty \subset \mathbb{R}^4$, see (4.1), where for each $k \geq 0$, the coefficient function $\Psi(k) = \Psi(k)(\xi,\underline{u})$ is a smooth field on $\mathbb{R}^2 \times (0,\infty)$ taking values in $\mathcal{R}$. By Proposition 5.3, the associated formal constraint field $[\,\Psi^\sharp\,]$ is itself a formal power series in $\frac{1}{u}$, that is, $[\,\Psi^\sharp\,](x) = \sum_{k=0}^{\infty}(\frac{1}{u})^k\,\Psi^\sharp(k)(\xi,\underline{u})$.

Remark 6.1. It also follows from Proposition 5.3 that, for each $k \geq 0$, the coefficient function $\Psi^\sharp(k)$ depends only on $\Psi(\ell)$, $0 \leq \ell \leq k$.

We shall construct all formal power series solution $[\,\Psi\,]$ on $\mathbf{Strip}_\infty \subset \mathbb{R}^4$ to **(SHS)** , more precisely to the system (5.7a), with initial conditions (5.3a) and (5.3b). We will use (5.7b) to show that the associated formal constraint field $[\,\Psi^\sharp\,]$ vanishes.

Definition 6.1. *Regard the components of $\Psi(k)$ and $\overline{\Psi(k)}$, $k \geq 0$, and their formal first coordinate derivatives, as an infinite family of independent abstract variables.*
Set $\mathcal{P}_0 = 0$. The generic symbol $\mathcal{P}_k$, $k \geq 1$, is an arbitrary polynomial in the components of $\Psi(\ell)$ and $\overline{\Psi(\ell)}$, $0 \leq \ell \leq k-1$, and all their first coordinate derivatives ($\frac{\partial}{\partial x^\mu}\Psi(\ell)$ and $\frac{\partial}{\partial x^\mu}\overline{\Psi(\ell)}$, $\mu = 1, 2, 3$), whose coefficients are (complex) polynomials in $\mathfrak{A}$, $\underline{u}$, $\mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda}_{a,\mathfrak{A}}$, $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$, and all their first coordinate derivatives. It is further required that the polynomial $\mathcal{P}_k$ have no constant term, that is, $\mathcal{P}_k$ vanishes when $\Psi(\ell)$ and $\frac{\partial}{\partial x^\mu}\Psi(\ell)$ vanish for all $0 \leq \ell \leq k-1$ and $\mu = 1, 2, 3$. We use the same symbol $\mathcal{P}_k$ for a vector or matrix all whose entries are polynomials of this kind.

Proposition 6.1. *Substitute $[\,\mathcal{M}_{a,\mathfrak{A}}\,]$ (see, Definition 5.2) for $\mathcal{M}_{a,\mathfrak{A}}$ and $[\,\Psi\,]$ (see,* (6.1)*) for Ψ in* (5.7a)*. Then $[\,\Psi\,]$ is a formal power series solution to* (5.7a) *if and only if its coefficients $\Psi(k)$, $k \geq 0$, satisfy a system of the form*

$$z_1(k) = \mathcal{P}_k \qquad k > 0 \qquad (6.2a)$$

$$z_2(k) = \mathcal{P}_k \qquad k > 0 \qquad (6.2b)$$

$$z_3(k) = \mathcal{P}_k \qquad k > 0 \qquad (6.2c)$$

$$\frac{\partial}{\partial \underline{u}}\, z_5(k) = \mathcal{P}_k \qquad k \geq 0 \qquad (6.2d)$$

$$(1-\delta_{k0}) z_4(k) = -\tfrac{2}{k-\delta_{k0}}\left(\mathbf{e}\,\tfrac{\partial}{\partial \overline{\xi}} + 2\boldsymbol{\lambda}\right) z_5(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2e)$$

$$\frac{\partial}{\partial \underline{u}}\, \omega_1(k) = -z_1(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2f)$$

$$\frac{\partial}{\partial \underline{u}}\, \omega_2(k) = -(2-\delta_{k0})\, \Re\big(\omega_1(0)\,\overline{\omega_1(k)}\,\big) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2g)$$

$$\frac{\partial}{\partial \underline{u}}\, \omega_3(k) = -z_2(k) - \overline{\boldsymbol{\lambda}}\, \omega_1(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2h)$$

$$\frac{\partial}{\partial \underline{u}}\, \omega_4(k) = -\boldsymbol{\lambda}\, \overline{\omega_1(k)} + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2i)$$

$$\omega_5(k) = -\tfrac{1}{k+1}\big(\omega_4(k) - \overline{\omega_3(k)}\,\big) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2j)$$

$$\omega_6(k) = \mathcal{P}_k \qquad k > 0 \qquad (6.2k)$$

$$\omega_7(k) = \mathcal{P}_k \qquad k > 0 \qquad (6.2l)$$

$$\frac{\partial}{\partial \underline{u}}\, \omega_8(k) = z_3(k) + 2i\Im\left(\boldsymbol{\lambda}\big(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\big)\right) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2m)$$

$$\frac{\partial}{\partial \underline{u}}\, f_1(k) = \quad \mathbf{e}\, \omega_1(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2n)$$

$$\frac{\partial}{\partial \underline{u}}\, f_2(k) = -i\,\mathbf{e}\, \omega_1(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2o)$$

$$f_3(k) = -\tfrac{2}{k+2}\, \Re\, \omega_8(k) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2p)$$

$$\frac{\partial}{\partial \underline{u}}\, f_4(k) = \quad 2\,\mathbf{e}\, \Re\left(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\right) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2q)$$

$$\frac{\partial}{\partial \underline{u}}\, f_5(k) = -2\,\mathbf{e}\, \Im\left(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\right) + \mathcal{P}_k \qquad k \geq 0 \qquad (6.2r)$$

Here, $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$. In (6.2g), (6.2k), (6.2p), (6.2q), (6.2r), *the generic symbol $\mathcal{P}_k$ is real valued when $\Psi(\ell)(\xi, \underline{u}) \in \mathcal{R}$ for all $0 \leq \ell \leq k$.*

Proof. Substitute the formal series (6.1) into the relevant / irrelevant form of system (5.7a) given in Proposition 5.2. Collect all coefficients of common powers of $\frac{1}{u}$. $\square$

Lemma 6.1. *For all* $a, \mathfrak{A} \neq 0$ *and all* **DATA**$(\xi, \underline{u})$ *as in Definition 5.1, there is a unique formal power series solution* (6.1) *to* (5.7a) *satisfying the initial conditions* (5.3a) *and* (5.3b) *in the sense of formal power series.*

Proof. The initial condition (5.3a) forces the zeroth coefficient function $\Psi(0)$ of (6.1) to be given exactly as in Definition 5.1. Observe that $\Psi(0)$ satisfies the $k = 0$ equations in (6.2). The coefficient functions $\Psi(k)$, $k \geq 1$, are constructed by induction. For each step k, equations (6.2a) to (6.2r) are solved exactly in this order to obtain $\Psi(k)$. The right hand side is explicitly known by induction and the "upper triangular" structure of (6.2a) to (6.2r). Whenever $\frac{\partial}{\partial \underline{u}}$ appears on the left hand side, it is inverted using the operator $\partial_{\underline{u}}^{-1}$ of Definition 5.1, because the constant of integration is zero by the initial condition (5.3b). By induction, one also verifies that $\Psi(k)$, $k \geq 0$, vanishes when $\underline{u} < \underline{u}_0$, so that (5.3b) is satisfied at all orders. It is essential at precisely this point that the generic polynomial $\mathcal{P}_k$ in Definition 6.1 has no constant term. Finally, by Proposition 6.1, there exists a formal power series solutions satisfying the hypothesis of the lemma. The construction given here is forced at every step, and therefore generates a unique formal power series. $\square$

Proposition 6.2. *For all* $a, \mathfrak{A} \neq 0$, *all* $\underline{u}_0 > 0$, *and all* **DATA**$(\xi, \underline{u})$ *as in Definition 5.1, there is a unique formal power series* $[\,\Psi\,]$ *on* $\mathbf{Strip}_\infty$, *which satisfies* (5.7a) *and* $[\,\Psi^\sharp\,] = 0$, *and which in addition satisfies* (5.3b) *and* $\omega_1(0) =$ **DATA**.
Moreover, for all $k \geq 0$, *the value of* $\Psi(k)$ *at* $(\xi, \underline{u}) \in \mathbb{R}^2 \times (0, \infty)$ *depends only on the restriction of* **DATA**$(\xi, \underline{u})$ *and its derivatives of all orders to the half-open line segment* $\{\xi\} \times (0, \underline{u}\,]$ *(formal finite speed of propagation).*

Proof. Suppose such a formal power series (6.1) exists. The condition $\omega_1(0) =$ **DATA**, the $k = 0$ equations in (6.2) and the initial condition (5.3b) taken together give the formulas in Definition 5.1 for all the components of $\Psi(0)$, apart from $\omega_7(0)$, $z_2(0)$, $z_3(0)$, $z_4(0)$ and $\omega_6(0)$. The remaining five formulas are exactly what is required for the vanishing of $p_5(0)$, $y_1(0)$, $y_2(0)$, $y_3(0)$ and $p_6(0)$, see (5.11c) and (5.11d). Here, $\Psi^\sharp(0) = (s(0), p(0), y(0))$. Therefore, this solution must coincide with the unique solution of Lemma 6.1, and gives the uniqueness statement of the proposition. Formal finite speed of propagation follows from an examination of the construction of $[\,\Psi\,]$ in the proof of Lemma 6.1.

To prove existence, we only have to show that $[\,\Psi^\sharp\,] = 0$. Note that

- $[\,\Psi^\sharp\,]$ is a formal power series solution to the linear homogeneous system (5.7b).
- $[\,\Psi^\sharp\,] = 0$ when $\underline{u} < \underline{u}_0$.
- $\Psi^\sharp(0) = 0$ on $\mathbb{R}^2 \times (0, \infty)$.

The first bullet follows from Proposition 2.3, because $[\,\Psi\,]$ is a formal power series solution to (5.7a). The second follows from (5.3b), which implies $[\,\Phi\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,]$ when $\underline{u} < \underline{u}_0$, and from $[\,\mathcal{M}^\sharp_{a,\mathfrak{A}}\,] = 0$. For the third, note that $p_5(0)$, $y_1(0)$, $y_2(0)$, $y_3(0)$ and $p_6(0)$ all vanish on $\mathbb{R}^2 \times (0, \infty)$ by the prescription of $\Psi(0)$ in Definition 5.1. By the first two bullets and by equation (5.12a), we conclude, step by step, that $s_1(0)$, $s_2(0)$, $p_1(0)$, $p_2(0)$, $p_3(0)$, $p_4(0)$, $s_4(0)$, $s_5(0)$, $p_7(0)$, $p_8(0)$ also vanish. The first equation in (5.12b) gives $s_3(0) = 0$. It remains to show that $p_9(0) = 0$ on $\mathbb{R}^2 \times (0, \infty)$. By (5.11c),

$$p_9(0) = -2\big(\mathbf{e}\,\tfrac{\partial}{\partial \bar{\xi}} + 2\,\boldsymbol{\lambda}\big)\,\omega_7(0) + 2\,\mathbf{e}\,\tfrac{\partial}{\partial \xi}\omega_6(0) + \omega_4(0) - \overline{\omega_3(0)}.$$

Definition 5.1 now implies $(\frac{\partial}{\partial \underline{u}})^2 p_9(0) = 0$. Therefore, $p_9(0) = 0$ on $\mathbb{R}^2 \times (0,\infty)$.

The three bullet statements imply, by induction on $k \geq 1$, that $\Psi^\sharp(k) = 0$ on the set $\mathbb{R}^2 \times (0,\infty)$. In fact, at each step k, one verifies, in the given order, that $y_1(k)$, $y_2(k)$, $y_3(k)$ all vanish by (5.12c), $p_1(k)$, $p_2(k)$, $p_3(k)$, $p_4(k)$ all vanish by (5.12a), $p_5(k)$, $p_6(k)$ both vanish by (5.12b), $p_7(k)$, $p_8(k)$ both vanish by (5.12c), $p_9(k)$, $s_3(k)$ both vanish by (5.12b), and $s_1(k)$, $s_2(k)$, $s_4(k)$, $s_5(k)$ all vanish by (5.12a). □

Proposition 6.3. *For all $k, R \geq 0$, all $0 < |\mathfrak{A}| \leq |a| \leq 1$, and all* **DATA**,

$$\|\Psi(k)\|_{C^R(\mathcal{Q})} \;\leq\; p_{k,R}\big(\|\mathbf{DATA}\|_{C^{R+2k+3}(\mathcal{Q})}\big) \qquad\qquad \mathcal{Q} = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0,2)$$

where $[\,\Psi\,]$ is the corresponding formal solution in Proposition 6.2, and $p_{k,R} : \mathbb{R} \to \mathbb{R}$, is an infinite family, indexed by $k, R \geq 0$, of universal polynomials without constant term. Here, $D_r(0)$ is the open disk of radius $r > 0$ in the (ξ^1, ξ^2)-plane.

Remark 6.2. The uniformity of the estimate in a, $\mathfrak{A}$, when $0 < |\mathfrak{A}| \leq |a| \leq 1$, will be exploited later. In particular, it is compatible with taking the limit $a = \mathfrak{A} \downarrow 0$.

Proof. Observe that:

- $\|\mathbf{e}_{a,\mathfrak{A}}\|_{C^R(\mathcal{Q})} \leq \frac{17}{2}$ and $\|\boldsymbol{\lambda}_{a,\mathfrak{A}}\|_{C^R(\mathcal{Q})} \leq \frac{17}{2}$ for all $R \geq 0$.
- $\|\partial_{\underline{u}}^{-1} g\|_{C^R(\mathcal{Q})} \leq 2\,\|g\|_{C^R(\mathcal{Q})}$ for all $R \geq 0$ and all functions $g(\xi, \underline{u})$ on $\mathcal{Q}$.

The existence of polynomials $p_{0,R}$, $R \geq 0$, follows by direct inspection of $\Psi_{a,\mathfrak{A}}(0)$, see Definition 5.1. The existence of polynomials $p_{k,R}$, $R \geq 0$, is shown by induction over $k \geq 0$. At each step $k \geq 1$, we use (6.2). By the inductive hypothesis and Definition 6.1 there is a polynomial $p'_{k,R}$ (depending only on k and R) so that each generic term $\mathcal{P}_k$ on the right hand sides of (6.2) satisfies $\|\mathcal{P}_k\|_{C^R(\mathcal{Q})} \leq p'_{k,R}\big(\|\mathbf{DATA}\|_{C^{R+2k+2}(\mathcal{Q})}\big)$. We can assume that $p'_{k,R}$ has no constant term, because $\mathcal{P}_k$ does not have one (see, Definition 6.1). Now, the existence of $p_{k,R}$, $R \geq 0$ follows directly from estimating the non generic terms on the right hand sides of (6.2a) to (6.2r), exploiting the upper triangular structure. Only in one equation, (6.2e), a coordinate derivative appears. □

Remark 6.3. Fix **DATA** and let $[\,\Psi_{a,\mathfrak{A}}\,]$ be the formal power series solution in Proposition 6.2. The indices have been added to make the dependence on $a, \mathfrak{A} \neq 0$ explicit. One can show, by induction, that $\Psi_{\mathfrak{A},\mathfrak{A}}(k)(\xi, \underline{u})$, $k \geq 0$, are polynomials in $\mathfrak{A}$. Just follow the construction of $[\,\Psi_{\mathfrak{A},\mathfrak{A}}\,]$ given in the proof of Lemma 6.1, and use the observation that $\mathbf{e}_{\mathfrak{A},\mathfrak{A}}$ and $\boldsymbol{\lambda}_{\mathfrak{A},\mathfrak{A}}$ are polynomials in $\mathfrak{A}$.

Let $\mathfrak{P}$ be the parity field symmetry, see Remark 2.9. Then $\mathfrak{P} \cdot [\,\Psi_{\mathfrak{A},\mathfrak{A}}\,] = [\,\Psi_{-\mathfrak{A},-\mathfrak{A}}\,]$. This is a direct consequence of $\mathfrak{P} \cdot \mathcal{M}_{\mathfrak{A},\mathfrak{A}} = \mathcal{M}_{-\mathfrak{A},-\mathfrak{A}}$, the uniqueness statement in Proposition 6.2 and the fact that $\omega_1(0)$ is $\mathfrak{P}$-even. Therefore, the $\mathfrak{P}$-even ($\mathfrak{P}$-odd) components of $\Psi_{\mathfrak{A},\mathfrak{A}}(k)$, $k \geq 0$, are even (odd) polynomials in $\mathfrak{A}$.

Let $[\,\Psi_{0,0}\,] = \lim_{\mathfrak{A}\downarrow 0} [\,\Psi_{\mathfrak{A},\mathfrak{A}}\,]$ (the limit is taken coefficient by coefficient). We have $[\,\Phi_{0,0}\,] = [\,\mathcal{M}_{0,0}\,] + u^{-M}[\,\Psi_{0,0}\,]$. The $\mathfrak{P}$-odd components all vanish. By inspection, the e_4, e_5 components also vanish (see, Remark 2.9). Therefore, $[\,\Phi_{0,0}\,]$ satisfies the hypothesis of Proposition 2.4, in the sense of formal power series. The field $[\,\widetilde{\Phi}_{0,0}\,]$ is a formal solution to **(subSHS)** , and $[\,\widetilde{\Phi}^\sharp_{0,0}\,] = 0$.

We now match constructions between two stereographic charts.

Proposition 6.4. *Choose $a, \mathfrak{A} \neq 0$. Pick* **DATA**$^\sigma(\xi, \underline{u})$ *as in Definition 5.1, for $\sigma = -, +$ and let $[\,\Psi^\sigma\,]$ be the associated solution in Proposition 6.2. The following statements are equivalent:*

- $\frac{|\xi|^2}{\xi^2}\mathbf{DATA}^\sigma\big(\frac{a}{\mathfrak{A}}\xi,\,\underline{u}\big) \;=\; \frac{\xi^2}{|\xi|^2}\,\mathbf{DATA}^{-\sigma}\big(\frac{a}{\mathfrak{A}}\,\frac{1}{\xi},\,\underline{u}\big)$ *when* $\xi \neq 0$.
- $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot [\,\Phi^\sigma\,] = [\,\Phi^{-\sigma}\,]$ *when* $\xi \neq 0$. *Here,* $[\,\Phi^\sigma\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,] + u^{-M}[\,\Psi^\sigma\,]$.
- $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot [\,\Psi^\sigma\,] = [\,\Psi^{-\sigma}\,]$ *when* $\xi \neq 0$.

Here, $-\sigma = +$ *when* $\sigma = -$*, and conversely,* $-\sigma = -$ *when* $\sigma = +$.

Proof. The equivalence of the last two bullets follows from Proposition 4.1, (b), and the fact that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ commutes with multiplication by u^{-M}. Each of the last two bullets implies the first. Just look at how $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ acts on the component ω_1. The first bullet implies the last two, because $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is a field symmetry, and by uniqueness in Proposition 6.2 (more precisely, by formal finite speed of propagation). $\square$

It is convenient (see Subsection 8.2) to make the

Definition 6.2. *For all* $(\xi, \underline{u}) \in \mathbb{R}^2 \times (0, \infty)$ *with* $\xi \neq 0$*, set*

$$\big(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathbf{DATA}\big)(\xi, \underline{u}) \;=\; \frac{\xi^2}{\bar{\xi}^2}\,\mathbf{DATA}\Big(\frac{a^2}{\mathfrak{A}^2}\,\frac{1}{\xi},\,\underline{u}\Big)$$

7. Energy Estimates

In this section, we prove an abstract local existence theorem for a general class of quasilinear symmetric hyperbolic systems, with a concrete breakdown criterion. Then, we develop appropriate energy estimates. These tools are applied in Section 8.

Convention 7.1. In this section,

$$\begin{aligned} x &= (x^1,\, x^2,\, x^3,\, x^4) = (\xi^1,\, \xi^2,\, \underline{u},\, u) \\ q &= (q^0,\, q^1,\, q^2,\, q^3) = (t,\, \xi^1,\, \xi^2,\, \underline{u}) \\ t &= u + \underline{u} \\ \mathbf{q} &= (q^1,\, q^2,\, q^3) = (\xi^1,\, \xi^2,\, \underline{u}) \end{aligned}$$

Let $D_r(\xi) \subset \mathbb{R}^2$ be the open disk of radius $r > 0$ around ξ and, generally, $B_r(p) \subset \mathbb{R}^N$ be the open ball of radius $r > 0$ around the point p.

For any parameter vector $a = (a_1, \ldots, a_k) \in (\mathbb{R}_+)^k$, the notation $X \lesssim_a Y$ signifies that $X \leq C\,Y$ for a constant $C = C(a) > 0$ that depends only on a. Dropping the subscript a, the notation $X \lesssim Y$ means $Y \leq C\,Y$ for a universal constant $C > 0$.

7.1. *Sobolev inequality.*

Lemma 7.1. *Let* $\partial_{\mathbf{q}} = (\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3})$*. If* $b \in (0, 2]$*, then for all* C^2*-functions* $f(\mathbf{q}) = f(q^1,\, q^2,\, q^3)$ *on the cylinder* $\mathbf{CYL} = D_{1/4}(0) \times (0, b) \subset \mathbb{R}^3$ *which vanish for* $q^3 < 1/4$*,*

$$\sup_{\mathbf{q}\,\in\,\mathbf{CYL}} |f(\mathbf{q})| \;\;\lesssim\;\; \Big(\sum_{|\alpha|=2} \|\partial_{\mathbf{q}}^\alpha f\|^2_{L^2(\mathbf{CYL})}\Big)^{1/2}, \qquad \alpha \in \mathbb{N}_0^3.$$

Proof. Let $B = D_{1/4}(0) \times \{0\}$ be the base of $\mathbf{CYL}$ and S^2 the unit sphere in $\mathbb{R}^3$. For each $\mathbf{q} \in \mathbf{CYL}$, let $\Gamma_{\mathbf{q}} \subset S^2$ be the set of all quotients $\boldsymbol{\zeta} = \frac{\mathbf{p}-\mathbf{q}}{|\mathbf{p}-\mathbf{q}|}$ where $\mathbf{p} \in B$. Set $l(\boldsymbol{\zeta}) = |\mathbf{p} - \mathbf{q}|$. We have

$$|f(\mathbf{q})| \;\leq\; \frac{1}{|\Gamma_{\mathbf{q}}|_{S^2}} \int_{\Gamma_{\mathbf{q}} \subset S^2} \mathrm{d}A_{S^2}(\boldsymbol{\zeta})\, F(\boldsymbol{\zeta})$$

where

$$F(\boldsymbol{\zeta}) \;=\; \int_0^{l(\boldsymbol{\zeta})} \mathrm{d}r\; r\, \Big| \Big\langle \boldsymbol{\zeta},\; \mathrm{H}(f)\big(\mathbf{q} + r\boldsymbol{\zeta}\big)\, \boldsymbol{\zeta} \Big\rangle \Big|$$

since, Taylor's theorem and the support properties of f imply $|f(\mathbf{q})| \leq F(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \Gamma_{\mathbf{q}}$. Here, $H(f)$ is the Hessian of f. Let $C_{\mathbf{q}} \subset \mathbb{R}^3$ be the convex hull of $B \cup \{\mathbf{q}\}$. By the Schwarz inequality,

$$|f(\mathbf{q})| \;\leq\; \frac{1}{|\Gamma_{\mathbf{q}}|_{S^2}} \Big(\int_{\Gamma_{\mathbf{q}} \subset S^2} \mathrm{d}A_{S^2}(\boldsymbol{\zeta}) \int_0^{l(\boldsymbol{\zeta})} 1\, \mathrm{d}r \Big)^{1/2} \Big(\int_{C_{\mathbf{q}}} \mathrm{d}^3\mathbf{y}\, |H(f)(\mathbf{y})|^2 \Big)^{1/2}$$

where, r^2 has disappeared into the measure $\mathrm{d}^3\mathbf{y}$ and $|M| = (\mathrm{tr}\, M^T M)^{1/2}$ is the Euclidean matrix norm. Also, observe that $l(\boldsymbol{\zeta}) \leq 3$ and $|\Gamma_{\mathbf{q}}|_{S^2}$ is bounded below by a universal constant, for instance $\pi/100$. By construction, $C_{\mathbf{q}} \subset \mathbf{CYL}$, and the proof is finished. $\square$

Lemma 7.2. *Let $b \in [1, 2]$. Then for all C^2-functions $f(\mathbf{q}) = f(q^1, q^2, q^3)$ on the cylinder $\mathbf{CYL} = D_{1/4}(0) \times (0, b) \subset \mathbb{R}^3$,*

$$\sup_{\mathbf{q} \in \mathbf{CYL}} |f(\mathbf{q})| \;\;\lesssim\;\; \Big(\sum_{|\alpha| \leq 2} \|\partial_{\mathbf{q}}^{\alpha} f\|^2_{L^2(\mathbf{CYL})} \Big)^{1/2}, \qquad \alpha \in \mathbb{N}_0^3. \tag{7.1}$$

Proof. By reflection symmetry, it suffices to show (7.1) when $\mathbf{q} \in \mathbf{CYL}$ satisfies $q^3 \geq \frac{b}{2} \geq \frac{1}{2}$. Fix a smooth transition function $\psi = \psi(q^3) : \mathbb{R} \to [0, 1]$ equal to 0 on $(-\infty, \frac{1}{4}]$ and equal to 1 on $[\frac{1}{2}, \infty)$. Then,

$$\begin{aligned} |f(\mathbf{q})|^2 = |(\psi f)(\mathbf{q})|^2 \;\; &\overset{\text{Lemma 7.1}}{\lesssim} \;\; \sum_{|\alpha|=2} \|\partial_{\mathbf{q}}^{\alpha}(\psi f)\|^2_{L^2(\mathbf{CYL})} \\ &\lesssim \;\; \|\psi\|^2_{C^2([0,1])} \sum_{|\alpha| \leq 2} \|\partial_{\mathbf{q}}^{\alpha} f\|^2_{L^2(\mathbf{CYL})} \qquad \square \end{aligned}$$

7.2. Finite speed of propagation for a general class of symmetric hyperbolic systems. We show finite speed of propagation in the context of the following hypotheses:

(FS0) $\mathcal{U} \subset \mathbb{R}^4$ is open and $\mathcal{A} = \mathcal{U} \times B_r(0)$ where $B_r(0) \subset \mathbb{R}^P$, $0 < r \leq +\infty$, $P \in \mathbb{N}$.
(FS1) $\mathbf{M}^{\mu}(q, \Theta)$, $\mu = 0, 1, 2, 3$, is a symmetric $P \times P$ matrix on $\mathcal{A}$. Moreover, $h(q, \Theta)$ is an $\mathbb{R}^P$ valued function on $\mathcal{A}$. Both $\mathbf{M}^{\mu}$ and h are C^1 on $\mathcal{A}$ and extend, with their derivatives, continuously to $\overline{\mathcal{A}}$, and $\mathbf{M}^0 > 0$ on $\overline{\mathcal{A}}$.

(FS2) Θ_1, Θ_2 are C^1-functions on $\mathcal{U}$ with values in $B_r(0) \subset \mathbb{R}^P$, that are solutions to the symmetric hyperbolic system

$$\mathbf{M}(q,\Theta)\Theta = h(q,\Theta), \qquad \mathbf{M} = \mathbf{M}^\mu \tfrac{\partial}{\partial q^\mu}. \tag{7.2}$$

Both Θ_1, Θ_2 extend, with their derivatives, continuously to $\overline{\mathcal{U}}$.

(FS3) $\mathcal{U}$ is either the set **Cone** or the intersection **Cone** $\cap$ ($\mathbb{R} \times$ **Half**).

Above, all the quantities are real. **Cone** is any set of the form

$$\big\{(t,\mathbf{q}) \in \mathbb{R}^4 :\ |\mathbf{q} - \mathbf{q}_0|_{\mathbb{R}^3} < v|t_1 - t|,\ t \in (t_0, t_1)\big\} \tag{7.3}$$

where $v > 0$, $\mathbf{q}_0 \in \mathbb{R}^3$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ are arbitrary, and **Half** is any open half-space in $\mathbb{R}^3$. We refer to v as the velocity of the set **Cone**.

Lemma 7.3. *Suppose* **(FS0)**, **(FS1)**, **(FS2)**, *then the difference $\Upsilon = \Theta_2 - \Theta_1$ satisfies the linear homogeneous symmetric hyperbolic system*

$$\mathbf{M}(q)\Upsilon = H(q)\Upsilon \tag{7.4}$$

where $\mathbf{M}(q) = \mathbf{M}(q,\Theta_1(q))$ and $H(q)$ is a square matrix. Moreover, $\mathbf{M}(q)$, its first derivatives and $H(q)$ are continuous on $\mathcal{U}$ and extend continuously to $\overline{\mathcal{U}}$.

Proof. Adding and subtracting,

$$\mathbf{M}(q,\Theta_1)\Upsilon = -\big(\mathbf{M}(q,\Theta_2) - \mathbf{M}(q,\Theta_1)\big)\Theta_2 + h(q,\Theta_2) - h(q,\Theta_1).$$

Set $\Theta_s = (1-s)\,\Theta_1 + s\,\Theta_2$, and

$$H_{ij}(q) = -\sum_k \Big(\int_0^1 \mathrm{d}s\, \frac{\partial(\mathbf{M}^\mu)_{ik}}{\partial\Theta^j}(q,\Theta_s(q))\Big)\frac{\partial\Theta_2^k}{\partial q^\mu}(q) + \int_0^1 \mathrm{d}s\, \frac{\partial h_i}{\partial\Theta^j}(q,\ \Theta_s(q)).$$

The proposition follows from the fundamental theorem of calculus. $\square$

Suppose **(FS3)**. In this case, let

$$\begin{aligned} S &= (\partial\mathcal{U}) \cap \big((t_0,t_1) \times \mathbb{R}^3\big) \\ B &= (\partial\mathcal{U}) \cap \big(\{t_0\} \times \mathbb{R}^3\big) \end{aligned}$$

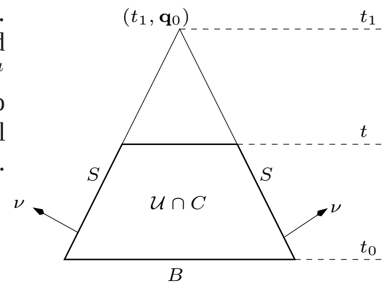
be the “lateral boundary” and “base” of $\mathcal{U}$. Note that S is a piecewise smooth hypersurface in $\mathbb{R}^4$. Let $\theta = \theta_\mu\,\mathrm{d}q^\mu$ be a smooth 1-form on the smooth components of S, such that $\theta(X) > 0$ for every vector X pointing *out* of $\mathcal{U}$.

Proposition 7.1. *Given* **(FS0)**, **(FS1)**, **(FS2)** *and* **(FS3)**, *the difference $\Upsilon = \Theta_2 - \Theta_1$ vanishes identically on $\overline{\mathcal{U}}$ when*

$$\Upsilon|_B \;=\; 0, \tag{7.5a}$$

$$\theta_\mu\,\mathbf{M}^\mu(q,\Theta_1(q)) \;\geq\; 0, \quad \textit{along the smooth components of } S. \tag{7.5b}$$

Proof. We abbreviate $\mathbf{M} = \mathbf{M}(q, \Theta_1(q))$. Define the C^1 "energy current" vector field $j^\mu = \Upsilon^T \mathbf{M}^\mu \Upsilon$ on $\mathcal{U}$. By hypothesis, j^μ and its derivatives extend continuously to $\overline{\mathcal{U}}$. For $t \in [t_0, t_1]$, let $E(t)$ be the integral of the component j^0 over $\mathcal{U} \cap (\{t\} \times \mathbb{R}^3)$. The Euclidean divergence theorem gives



$$E(t) = \int_{\mathcal{U}\cap C} \partial_\mu j^\mu - \int_{S \cap C} \langle j, \nu \rangle$$

where $C = (t_0, t) \times \mathbb{R}^3$, because, by (7.5a), $E(t_0) = 0$. For $\mathcal{U} = \mathbf{Cone}$, see the nearby figure. By (7.5b), the outward flux $\int_{S\cap C} \langle j, \nu \rangle$ of j through $S \cap C$ is positive. By construction, $\partial_\mu j^\mu = \Upsilon^T K \Upsilon$, where $K = \partial_\mu \mathbf{M}^\mu + H^T + H$, since $\mathbf{M}^\mu$ is symmetric, and Υ is a solution to (7.4). By Proposition 7.3, K is continuous on $\overline{\mathcal{U}}$ and therefore bounded. Also, there is a constant $k > 0$ such that $|\Upsilon^T K \Upsilon| \le k j^0$, because $\mathbf{M}^0$ is strictly uniformly positive definite on the compact set $\overline{\mathcal{U}}$. It follows that

$$E(t) \,\le\, \int_{\mathcal{U}\cap C} |\Upsilon^T K \Upsilon| \,\le\, k \int_{t_0}^{t} \mathrm{d}s\, E(s),$$

for all $t \in [t_0, t_1]$. Consequently, $E(t) = 0$ on the interval $[t_0, t_1]$. In other words, $\Upsilon = 0$. $\square$

Proposition 7.2. *Let $\mathcal{U} = (t_0, t_1) \times \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^3$ is open, and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Suppose* **(FS0)**, **(FS1)**, **(FS2)** *and let $\Upsilon = \Theta_2 - \Theta_1$. Further suppose*

(i) $\Upsilon|_{\{t_0\}\times\mathcal{X}} = 0$.

(ii) $\mathbf{M}^0 \ge a$ *and* $-b \le \mathbf{M}^i \le b,\ i = 1, 2, 3$, *on* $\overline{\mathcal{A}}$, *for constants* $a, b > 0$.

Part 1. *Set $v^* = \sqrt{3}\,(b/a) > 0$. Then Υ vanishes at $(t, \mathbf{q}) \in \mathcal{U}$ if*

$$\mathrm{dist}_{\mathbb{R}^3}\big(\mathbb{R}^3 \setminus \mathcal{X},\, \mathbf{q}\big) \,>\, v^* \,|t_1 - t_0|. \tag{7.6}$$

Part 2. *If, in addition, $\mathbf{M}^3 \ge 0$ on $\overline{\mathcal{A}}$, then Υ vanishes at $(t, \mathbf{q}) \in \mathcal{U}$ if*

$$\mathrm{dist}_{\mathbb{R}^3}\Big(\overline{\mathbf{Half}_\mathbf{q}} \cap \big(\mathbb{R}^3 \setminus \mathcal{X}\big),\, \mathbf{q}\Big) \,>\, v^* \,|t_1 - t_0| \tag{7.7}$$

where $\mathbf{Half}_\mathbf{q} = \{\mathbf{y} \in \mathbb{R}^3 \,|\, y^3 < q^3\}$.

Proof (of Part 1). Let $\mathbf{q} \in \mathcal{X}$ satisfy (7.6). Observe that, for the set $\mathbf{Cone} \subset \mathcal{U}$ with velocity v^*, base at time t_0 and vertex at $(t_1, \mathbf{q})$, we have

$$\theta_\mu\, \mathbf{M}^\mu|_{S\times B_r(0)} \ge 0 \tag{7.8}$$

by the choice of v^* and (ii). Here, the lateral boundary S and the 1-form θ are just as in the discussion above Proposition 7.1. We can now apply Proposition 7.1 to **Cone**, and conclude $\Upsilon|_{\overline{\mathbf{Cone}}} = 0$. $\square$

Proof (of Part 2). Suppose that $\mathbf{M}^3 \ge 0$ on $\overline{\mathcal{A}}$. Let $\mathbf{q} \in \mathcal{X}$ satisfy (7.7). The boundary of the set $\mathbf{Cone} \cap \mathbf{Half}_\mathbf{q} \subset \mathcal{U}$, where **Cone** has velocity v^*, base at time t_0 and vertex at $(t_1, \mathbf{q})$, has two smooth components. On the "round" one, the inequality (7.8) holds again by the choice of v^*, and on the "flat" one by $\theta_\mu\, \mathbf{M}^\mu = \mathbf{M}^3 \ge 0$ for θ proportional to $(0, 0, 1, 0)$. We have $\Upsilon|_{\overline{\mathbf{Cone}\cap\mathbf{Half}_\mathbf{q}}} = 0$, by Proposition 7.1. $\square$

7.3. Existence / breakdown theorem.
Assumptions for the Existence / breakdown theorem: All the quantities here are real.

(EB0) Let $\mathcal{U} = (-\infty, T) \times \mathbb{R}^3$, where $T \in \mathbb{R}$ and let $\mathcal{A} = \mathcal{U} \times B_2(0) \subset \mathbb{R}^4 \times \mathbb{R}^P$.
(EB1) $\mathbf{M}^\mu(q, \Theta)$ is a symmetric $P \times P$ matrix on $\mathcal{A}$. Particularly, $\mathbf{M}^0 \geq \frac{1}{2}$.
(EB2) $h(q, \Theta)$ is an $\mathbb{R}^P$ valued function on $\mathcal{A}$.
(EB3) Both h and $\mathbf{M}^\mu$ are smooth on $\mathcal{A}$ and their derivatives of all orders extend continuously to $\overline{\mathcal{A}}$.
(EB4) $\mathcal{K} \subset \mathcal{Q} \subset \mathbb{R}^3$ with $\mathcal{K}$ compact, $\mathcal{Q}$ open, such that on $(-\infty, T) \times (\mathbb{R}^3 \backslash \mathcal{K}) \times B_2(0)$, the matrix $\mathbf{M}^\mu$ is constant and denoted by $\not{\mathbf{M}}^\mu$ and $h(q, \Theta) = \not{H}(t)\, \Theta$, where $\not{H}(t)$ is a matrix depending only on $t = q^0$. It is assumed, $\not{\mathbf{M}}^0 \geq \frac{1}{2}$. We can naturally extend $\mathbf{M}^\mu$ and h, by $\not{\mathbf{M}}^\mu$ and $\not{H}(t)\Theta$, to $(-\infty, T) \times (\mathbb{R}^3 \setminus \mathcal{K}) \times \mathbb{R}^P$.

We now formulate and prove an existence theorem for the quasilinear symmetric hyperbolic system

$$\mathbf{M}(q, \Theta)\, \Theta = h(q, \Theta) \qquad \mathbf{M} = \mathbf{M}^\mu \tfrac{\partial}{\partial q^\mu}. \tag{7.9}$$

Proposition 7.3. *Suppose* **(EB0)**, **(EB1)**, **(EB2)**, **(EB3)**, **(EB4)**.
Part 1. *For each $t_0 < T$, there is a $t_1 \in (t_0, T]$ and a smooth solution $\Theta : [t_0, t_1) \times \mathbb{R}^3 \to \mathbb{R}^P$ of* (7.9) *with trivial initial data, $\Theta(t_0, \,\cdot\,) = 0$, such that* $\operatorname{supp} \Theta \subset [t_0, t_1) \times B_r(0)$ *for some finite $r > 0$, and*

$$\Theta\big([t_0, t_1) \times \mathcal{Q}\big) \subset B_1(0) \subset \mathbb{R}^P \tag{7.10}$$

and such that $t_1 \neq T$ implies either one or both of:

$(\mathbf{Break})_1$*:* $\overline{\Theta([t_0, t_1) \times \mathcal{Q})} \not\subset B_1(0) \subset \mathbb{R}^P$.
$(\mathbf{Break})_2$*:* *The vector field $\partial_q \Theta$ is unbounded on $[t_0, t_1) \times \mathcal{Q}$.*

Part 2. *Suppose in addition that $\mathbf{M}^3 \geq 0$ on $\mathcal{A}$, and $h(q, 0) = 0$ when $q^3 < \frac{1}{2}$. Then the solution Θ of Part 1 vanishes identically for $q^3 < \frac{1}{2}$.*

Proof (of Part 1). Fix a smooth transition function $\psi = \psi(|\Theta|) : \mathbb{R} \to [0, 1]$ which is equal to 1 on $(-\infty, \frac{4}{3})$ and equal to 0 on $(\frac{5}{3}, \infty)$. It is for this reason that $B_2(0)$ appears in **(EB0)**. Set

$$\mathbf{N} = \psi\, \mathbf{M} + (1 - \psi)\, \not{\mathbf{M}}, \qquad g = \psi\, h + (1 - \psi)\, \not{H}\, \Theta.$$

By construction, g and the symmetric matrix $\mathbf{N}^\mu$ are smooth on $\mathcal{B} = \mathcal{U} \times \mathbb{R}^P$, and their derivatives of all orders extend continuously to $\overline{\mathcal{B}}$. Note that $\mathbf{N}^0 \geq \frac{1}{2}$ on $\mathcal{B}$ and there is a constant $b > 0$ such that $-b \leq \mathbf{N}^i \leq b$, $i = 1, 2, 3$ on $[t_0, T] \times \mathbb{R}^3 \times \mathbb{R}^P$. The latter statement follows from the fact that $\mathbf{N}^\mu$ is constant ($= \not{\mathbf{M}}^\mu$) on the complement, in $[t_0, T] \times \mathbb{R}^3 \times \mathbb{R}^P$, of the compact set $[t_0, T] \times \mathcal{K} \times \overline{B_2(0)}$. Fix the velocity $v^* = 2\sqrt{3}\, b$ (see Proposition 7.2).

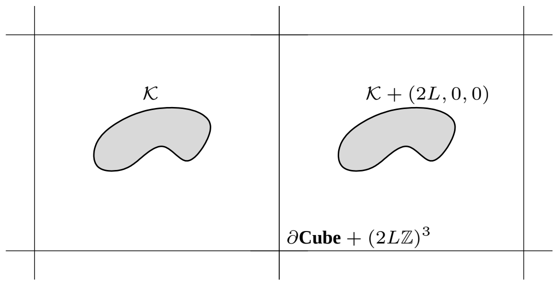
We want to reduce our existence / breakdown theorem to [Tay]. To do this, fix $L > 0$ big enough so that

$$\begin{aligned} \mathcal{K} \;\subset\; \textbf{Cube} \;&\stackrel{\text{def}}{=}\; [-L, L]^3 \\ \operatorname{dist}_{\mathbb{R}^3}\big(\partial\, \textbf{Cube},\, \mathcal{K}\big) \;&>\; 1 + v^*\, |T - t_0| \end{aligned}$$

and smoothly extend $\mathbf{N}$ and $\mathbf{g}$ from $(-\infty, T) \times \textbf{Cube} \times \mathbb{R}^P$ to spatially periodic matrix and vector valued functions on $(-\infty, T) \times \mathbb{R}^3 \times \mathbb{R}^P$. With these preliminaries, the hypotheses of Proposition 2.1 on page 370, Proposition 1.5 on page 365 and Corollary 1.6 on page 366 in [Tay] are all satisfied, and there is a $\tau \in (t_0, T]$ and a spatially periodic smooth solution $\Theta : [t_0, \tau) \times \mathbb{R}^3 \to \mathbb{R}^P$ with trivial initial data at $t = t_0$ to the symmetric hyperbolic system corresponding to the spatially periodic extension of $\mathbf{N}$ and $\mathbf{g}$, such that, if $\tau \neq T$, then the vector

$$\big(\Theta,\, \partial_q\, \Theta\big) \;\in\; \mathbb{R}^P \oplus \big(\mathbb{R}^4 \otimes \mathbb{R}^P\big)$$

is unbounded on $[t_0, \tau) \times \mathbb{R}^3$. (There is one caveat: [Tay], for convenience, considers systems defined for all time. By direct inspection, his argument applies to any open subinterval of $\mathbb{R}$.)



Let $\mathcal{J} = \mathcal{K} + (2L\mathbb{Z})^3$. By construction, the spatially periodic system introduced in the last paragraph reduces to $\not{\mathbf{M}}\Theta = \not{H}(t)\Theta$, on $(-\infty, T) \times (\mathbb{R}^3 \setminus \mathcal{J}) \times \mathbb{R}^P$, and admits the trivial solution. Intuitively, “signals can travel at most a distance $v^*|T - t_0|$”, which is less than the distance between $\mathcal{K}$ and $\partial\,\textbf{Cube}$. This intuition is formalized by applying Proposition 7.2 to the open set $(t_0, \tau - \epsilon) \times (\mathbb{R}^3 \setminus \mathcal{J})$ for arbitrarily small $\epsilon > 0$. Consequently, Θ vanishes at every point $(t, \mathbf{q}) \in [t_0, \tau) \times \mathbb{R}^3$ with $\mathrm{dist}_{\mathbb{R}^3}(\mathcal{J},\, \mathbf{q}) > \frac{1}{2} + v^*|T - t_0|$, because $\Theta|_{t=t_0} = 0$. It follows from our choice of L that the periodic solution Θ vanishes in a neighborhood of $[t_0, \tau) \times (\partial\,\textbf{Cube})$. For this reason, and because of **(EB4)**, the modified field

$$[t_0, \tau) \times \mathbb{R}^3 \;\ni\; q \;\;\mapsto\; \begin{cases} 0, & \text{if } \mathbf{q} \in (\mathbb{R}^3 \setminus \textbf{Cube}) \\ \Theta(q), & \text{if } \mathbf{q} \in \textbf{Cube} \end{cases} \qquad \in\; \mathbb{R}^P \tag{7.11}$$

which we continue to call Θ, is a smooth solution to the non-periodic system $\mathbf{N}\Theta = g$ with trivial initial data. Moreover, if $\tau \neq T$, then the vector $(\Theta,\, \partial_q\, \Theta)$ is unbounded.

Suppose $\tau \neq T$. We show that $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V} = [t_0, \tau) \times (\mathbb{R}^3 \setminus \mathcal{Q})$, and consequently, unbounded on $[t_0, \tau) \times \mathcal{Q}$. For any time $t_2 \in (t_0, \tau)$, decompose

$$\begin{aligned} [t_0, \tau) \times \mathbb{R}^3 &= \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \\ \mathcal{V}_1 &= [t_0, \tau) \times (\mathbb{R}^3 \setminus \textbf{Cube}) \\ \mathcal{V}_2 &= [t_0, t_2] \times \textbf{Cube} \\ \mathcal{V}_3 &= (t_2, \tau) \times \textbf{Cube} \end{aligned}$$

By (7.11), the vector $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V} \cap \mathcal{V}_1$ and, by compactness, also on $\mathcal{V} \cap \mathcal{V}_2$.

Let
$$d \;=\; \mathrm{dist}_{\mathbb{R}^3}(\mathcal{K},\, \mathbb{R}^3 \setminus \mathcal{Q}) \;>\; 0.$$
To verify that $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V}\cap\mathcal{V}_3$, we choose $t_2 \in (t_0, \tau)$ with $v^*\, |\tau - t_2| \leq d/2$. Intuitively, this choice means that "signals can travel at most a distance $d/2$ in the time interval (t_2, τ)", which is less than the distance between $\mathcal{K}$ and $\mathbb{R}^3 \setminus \mathcal{Q}$. There exists (see, for instance, [Tay2]) a unique smooth solution $\Theta_2 : [t_2, T) \times \mathbb{R}^3 \to \mathbb{R}^P$ to the linear system $\not{\mathbf{M}}\Theta_2 = \not{H}(t)\Theta_2$ with initial condition $(\Theta - \Theta_2)|_{t=t_2} = 0$. In particular, $(\Theta_2,\, \partial_q\, \Theta_2)$ is bounded on the compact set $\overline{\mathcal{V}\cap\mathcal{V}_3}$. By **(EB4)**, Θ is a solution to the same system on $[t_0, \tau) \times (\mathbb{R}^3 \setminus \mathcal{K})$. As in the last paragraph, Proposition 7.2 implies that $\Theta = \Theta_2$ on $[t_2, \tau) \times (\mathbb{R}^3 \setminus \mathcal{Q}) \;\supset\; \mathcal{V}\cap\mathcal{V}_3$. Consequently, $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V}\cap\mathcal{V}_3$, and we are done.

The final step is to remove the transition function ψ. Let
$$\mathcal{I} = \Big\{ t \in [t_0, \tau) \;\Big|\; \overline{\Theta\big([t_0, t] \times \mathcal{Q}\big)} \subset B_1(0) \subset \mathbb{R}^P \Big\}.$$
We show that $\mathcal{I} = [t_0, t_1)$, where $t_1 \in (t_0, \tau]$. First, $t_0 \in \mathcal{I}$ since the initial data vanishes. Second, if $t' \in \mathcal{I}$, then $[t_0, t'] \in \mathcal{I}$. Let $t_1 = \sup \mathcal{I}$. If $t_1 = \tau$, then $t_1 \neq \mathcal{I}$. If $t_1 < \tau$, assume by contradiction $t_1 \in \mathcal{I}$. Then, the compact set $\overline{\Theta([t_0, t_1] \times \mathcal{Q})}$ is contained in a ball $B_r(0) \subset \mathbb{R}^P$ of radius $r < 1$. However, $\partial_t \Theta$ is bounded on $[t_0, t_1] \times \mathbb{R}^3$, since $\mathrm{supp}_{\mathbb{R}^3}(\partial_t \Theta)(t,\, \cdot\,) \subset$ **Cube** is compact for all $t \in [t_0, t_1]$. Therefore, $t_1 + \epsilon \in \mathcal{I}$ for all sufficiently small $\epsilon > 0$.

The smooth solution of (7.9) that we are looking for is $\Theta|_{[t_0, t_1) \times \mathbb{R}^3}$. Indeed, it has trivial initial data, support contained in $[t_0, t_1) \times$ **Cube** and satisfies (7.10). If $q \in [t_0, t_1) \times \mathcal{Q}$, then $|\Theta(q)| \leq 1$ and $\psi(|\Theta(q)|) = 1$, by the definition of t_1. In this case, the system $\mathbf{N}\Theta = g$ reduces to (7.9). On $[t_0, t_1) \times (\mathbb{R}^3 \setminus \mathcal{K})$, **(EB4)** directly implies that the system also reduces to (7.9).

If $t_1 \neq T$, there are two alternatives: $t_1 < \tau \leq T$ and $t_1 = \tau < T$. For the first, we use the continuity of Θ on $\overline{[t_0, t_1] \times \mathcal{Q}}$ to conclude that
$$\overline{\Theta([t_0, t_1) \times \mathcal{Q})} \;=\; \overline{\Theta([t_0, t_1] \times \mathcal{Q})} \;\not\subset\; B_1(0)$$
since $t_1 \notin \mathcal{I}$. That is, we have $(\textbf{Break})_1$. For the second, $\tau \neq T$, and $(\Theta,\, \partial_q\, \Theta)$ is unbounded on $[t_0, t_1) \times \mathcal{Q}$. Since Θ is bounded, $(\textbf{Break})_2$ applies. The proof of Part 1 is complete. $\square$

Proof (of Part 2). Let $\mathbf{Half} = \{\mathbf{q} \in \mathbb{R}^3 \,|\, q^3 < \frac{1}{2}\}$. The assumption $h(q, 0) = 0$, when $q^3 < \frac{1}{2}$, implies that $\Theta_1 = 0$ is a solution to $\mathbf{N}\Theta = g$ on $(t_0, t_1) \times$ **Half**. Also, $\Theta_2 = \Theta$ is a smooth solution, and $(\Theta_2 - \Theta_1)|_{\{t_0\} \times \mathbf{Half}} = 0$. The assumption $\mathbf{M}^3 \geq 0$ on $\mathcal{A}$ implies $\not{\mathbf{M}}^3 \geq 0$ and consequently, $\mathbf{N}^3 \geq 0$ on $\mathcal{U} \times \mathbb{R}^P$. At last, Part 2 of Proposition 7.2, applied to the open set $(t_0, t_1 - \epsilon) \times$ **Half** with arbitrarily small $\epsilon > 0$, forces $\Theta_2 - \Theta_1 = \Theta$ to vanish on $[t_0, t_1) \times$ **Half**. $\square$

7.4. Energy Estimate.
Assumptions for the energy estimate: All the quantities here are real.

(E0) $\mathcal{U} = \mathcal{I} \times \mathcal{O}(b)$ where $\mathcal{O}(b) = \mathbb{R}^2 \times (0, b) \subset \mathbb{R}^3$, $\;b \in [1, 2]$ and $\mathcal{I} = (t_0, t^*)$, $-\infty < t_0 < t^* \leq -1$.

(E1) $\mathbf{M}^\mu(q)$ is a symmetric $P\times P$ matrix on $\mathcal{U}$. Particularly, for all $q\in\mathcal{U}$, $\frac{1}{2}\leq\mathbf{M}^0\leq 2$ and $\mathbf{M}^3\geq 0$. We assume the integer P is less than some big absolute constant, say $P\leq 10^9$.
(E2) $H(q)$ is a $P\times P$ matrix on $\mathcal{U}$.
(E3) $\mathbf{Src}(q)$ is an $\mathbb{R}^P$ valued function on $\mathcal{U}$.
(E4) $\Theta(q)$ is an $\mathbb{R}^P$ valued function on $\mathcal{U}$, which is a solution to the linear, inhomogeneous, symmetric hyperbolic system

$$\mathbf{M}(q)\Theta = H(q)\Theta + \mathbf{Src}(q), \qquad \mathbf{M}=\mathbf{M}^\mu\tfrac{\partial}{\partial q^\mu}. \tag{7.12}$$

(E5) Fix a non-negative integer R. Then, $\mathbf{M}^\mu$, Θ (resp. H, $\mathbf{Src}$) are C^{R+1} (resp. C^R) functions on $\mathcal{U}$, all of whose derivatives of order $\leq R+1$ (resp. $\leq R$) extend continuously to $\overline{\mathcal{U}}$.
(E6) $\operatorname{supp}\Theta(t,\,\cdot\,)$ and $\operatorname{supp}\mathbf{Src}(t,\,\cdot\,)$ are contained in a ball in $\mathbb{R}^3$ with radius independent of t. Moreover, Θ and $\mathbf{Src}$ vanish identically when $q^3<\frac{1}{2}$.

Let $\mathbb{R}^P=\mathbb{R}^{P_1}\oplus\mathbb{R}^{P_2}\oplus\mathbb{R}^{P_3}$. We decompose

$$\Theta=(\Theta_1,\,\Theta_2,\,\Theta_3), \qquad\qquad \mathbf{Src}=(\mathbf{Src}_1,\,\mathbf{Src}_2,\,\mathbf{Src}_3).$$

Each $P\times P$ matrix is decomposed into nine blocks of size $P_m\times P_n$, where $m,n=1,2,3$. Especially,

$$\mathbf{M}^\mu=(M^\mu_{mn})_{m,n=1,2,3}, \qquad\qquad H=(H_{mn})_{m,n=1,2,3}. \tag{7.13}$$

(E7) The matrix $\mathbf{M}^\mu$ is block-diagonal, $\mathbf{M}^\mu=\operatorname{diag}(M^\mu_1,\,M^\mu_2,\,M^\mu_3)$, and $(M_2)_{ij}=\mu(q)\,\delta_{ij}\,(\frac{\partial}{\partial q^0}+\frac{\partial}{\partial q^3})$, $i,j=1,\ldots,P_2$, for some function μ.
(E8) $\not{\mathbf{M}}^\mu$ are constant symmetric $P\times P$ matrices, with $\frac{1}{2}\leq\not{\mathbf{M}}^0\leq 2$, $\not{\mathbf{M}}^1=0$, $\not{\mathbf{M}}^2=0$ and $\not{\mathbf{M}}^3\geq 0$.
(E9) $\not{H}(t)$ is a $P\times P$ matrix depending only on t with $\mathbb{R}^{P_1}\oplus\mathbb{R}^{P_2}\oplus\mathbb{R}^{P_3}$ block-form

$$\not{H}=\big(\not{H}_{mn}\big)_{m,n=1,2,3}=\begin{pmatrix}0&0&0\\ \not{H}_1&0&0\\ 0&|t|^{-1}\not{H}_2&|t|^{-1}\not{H}_3\end{pmatrix}$$

where $\not{H}_1$, $\not{H}_2$, $\not{H}_3$ are constant matrices and $\not{H}_3\leq 0$.

Definition 7.1. *For every open $\mathcal{X}\subset\mathbb{R}^3$, the energy of f contained in $\mathcal{X}$ at time t is*

$$E^k_{\mathcal{X}}\{f\}(t)\overset{def}{=}\sum_{\substack{|\alpha|\leq k\\ \alpha\in\mathbb{N}_0^4}}\int_{\mathcal{X}}\mathrm{d}^3\mathbf{q}\,|\partial^\alpha f(t,\mathbf{q})|^2 \tag{7.14}$$

and the supremum norm

$$\mathbf{Sup}^{(k)}_{\mathcal{X}}\{f\}(t)\overset{def}{=}\sup_{\substack{|\alpha|\leq k\\ \alpha\in\mathbb{N}_0^4}}\ \sup_{\mathbf{q}\in\mathcal{X}}|\partial^\alpha f(t,\mathbf{q})| \tag{7.15}$$

for any scalar, vector or matrix valued C^k-function f. As usual, we denote $\partial^\alpha=\prod_{\mu=0}^3(\partial_\mu)^{\alpha_\mu}$ where $\partial_\mu=\frac{\partial}{\partial q^\mu}$, for any $\alpha=(\alpha_\mu)_{\mu=0,1,2,3}\in\mathbb{N}_0^4$. The pointwise norm $|\cdot|$ is always the Euclidean norm (for matrices, $|A|^2=\operatorname{tr}(A^TA)$).

(E10) There are constants $\mathbf{c}_1 \geq 0$ and $J > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|^{2J+2}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_1\}(t) \\ |t|^{2J}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_2\}(t) \\ |t|^{2J+2}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_3\}(t) \end{array}\right\} \leq \mathbf{c}_1^2,$$

(E11a) Assume $R \geq 4$. There is a constant $\mathbf{c}_2 > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|^2\, E^R_{\mathcal{O}(b)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}(t) \\ |t|^2\, E^R_{\mathcal{O}(b)}\{H_{1n} - \not{H}_{1n}\}(t) \\ E^R_{\mathcal{O}(b)}\{H_{2n} - \not{H}_{2n}\}(t) \\ |t|^2\, E^R_{\mathcal{O}(b)}\{H_{3n} - \not{H}_{3n}\}(t) \end{array}\right\} \leq \mathbf{c}_2^2.$$

(E11b) Assume $R \geq 0$. There is a constant $\mathbf{c}_2 > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|\, \mathbf{Sup}^{(\max\{1,R\})}_{\mathcal{O}(b)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}(t) \\ |t|\, \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{1n} - \not{H}_{1n}\}(t) \\ \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{2n} - \not{H}_{2n}\}(t) \\ |t|\, \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{3n} - \not{H}_{3n}\}(t) \end{array}\right\} \leq \mathbf{c}_2.$$

Proposition 7.4 (Energy Estimate). *Suppose the hypotheses* **(E0)** *through* **(E10)** *hold, and, also, either* **(E11a)** *or* **(E11b)** *holds. Let $J_0 > 0$ and assume $J \geq J_0$, see* **(E10)**. *Then, there are constants $\mathbf{c}_3(X) \in (0,1)$, $\mathbf{c}_4(X) > 0$ depending only on $X = \big(R, J_0, |\not{H}_1|, |\not{H}_2|, |\not{H}_3|\big)$, such that $\mathbf{c}_2 \leq \mathbf{c}_3(X)$ and $|t^*|^{-1} \leq \mathbf{c}_3(X)$ imply that*

$$\sqrt{E^R_{\mathcal{O}(b)}\{\Theta\}(\tau)} \leq \mathbf{c}_4(X)\, \frac{|t_0|^J \sqrt{E^R_{\mathcal{O}(b)}\{\Theta\}(t_0)} + \mathbf{c}_1}{|\tau|^J} \tag{7.16}$$

for all $\tau \in \mathcal{I}$ (see, **(E0)** *for the definition of $\mathcal{I}$).*

Proof. In the proof, we denote $E^R = E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$.

Preliminaries 1: For a function f with values in $\mathbb{R}^{Pi}$, $i = 1, 2, 3$, we define the energy naturally associated to the linear symmetric hyperbolic system (7.12)

$$\mathbf{E}^0_i\{f\}(t) = \int_{\mathcal{O}(b)} \mathrm{d}^3\mathbf{q}\, \big(f^T M^0_i f\big)(t, \mathbf{q}) \quad , \quad \mathbf{E}^R_i\{f\}(t) = \sum_{\substack{|\alpha| \leq R \\ \alpha \in \mathbb{N}_0^4}} \mathbf{E}^0_i\{\partial^\alpha f\}(t) \tag{7.17}$$

See, (7.13). This energy is comparable, by **(E1)**, to the one defined in (7.14). Namely,

$$E^R\{f\}(t) \leq 2\, \mathbf{E}^R_i\{f\}(t), \qquad \mathbf{E}^R_i\{f\}(t) \leq 2\, E^R\{f\}(t). \tag{7.18}$$

If $R \geq 2$ and f is a vector or matrix valued C^R function, Lemma 7.2 implies:

$$\mathbf{Sup}^{(R-2)}\{f\}(t) \lesssim_R \sqrt{E^R\{f\}(t)}. \tag{7.19}$$

If $\alpha \in \mathbb{N}_0^4$, $|\alpha| \leq R$ and $R \geq 4$ (to be used in the case **(E11a)**) implies

$$E^R\{f_1 f_2\}(t) \lesssim_R E^R\{f_1\}(t)\, E^R\{f_2\}(t) \tag{7.20}$$
$$E^0\big\{\, [\partial^\alpha, f_1] f_2 \,\big\}(t) \lesssim_R \sum_{|\beta|=1} E^{R-1}\{\partial^\beta f_1\}(t)\, E^{R-1}\{f_2\}(t) \tag{7.21}$$

whereas $R \geq 0$ (to be used in the case **(E11b)**) implies

$$E^R\{f_1 f_2\}(t) \lesssim_R \big(\textbf{Sup}^{(R)}\{f_1\}(t)\big)^2\, E^R\{f_2\}(t) \tag{7.22}$$
$$E^0\big\{\, [\partial^\alpha, f_1] f_2 \,\big\}(t) \lesssim_R \sum_{|\beta|=1} \big(\textbf{Sup}^{(R-1)}\{\partial^\beta f_1\}(t)\big)^2\, E^{R-1}\{f_2\}(t) \tag{7.23}$$

Inequalities (7.22) and (7.23) are direct consequences of the product rule. The inequalities (7.20) and (7.21), require, in addition, $R \geq 4$ and the Sobolev inequality (7.19). In fact, by the product rule,

$$E^R\{f_1 f_2\}(t) \lesssim_R \sum_{|\alpha|+|\beta| \leq R} \int_{\mathcal{O}(b)} \mathrm{d}^3\mathbf{q}\, \big|\partial^\alpha f_1(t,\mathbf{q})\big|^2\, \big|\partial^\beta f_2(t,\mathbf{q})\big|^2$$

For each pair of multiindices (α, β) with $|\alpha| + |\beta| \leq R$, at least one of $|\alpha|$ or $|\beta|$ is less than or equal to $R-2$, say α. Then, by the Sobolev inequality,

$$\sup_{\mathbf{q} \in \mathcal{O}(b)} |\partial^\alpha f_1(t,\mathbf{q})|^2 \leq E^R\{f_1\}(t)$$

Inequality (7.20) follows at once. This argument works for $R \geq 3$. An entirely similar argument gives (7.21), but with $R \geq 4$.

Preliminaries 2: In this subsection, $t \in \mathcal{I}$ and $\alpha \in \mathbb{N}_0^4$, $|\alpha| \leq R$, are arbitrary. We apply ∂^α to (7.12) and obtain (all the derivatives make sense classically)

$$\mathbf{M}(\partial^\alpha \Theta) = \not{H}\, \partial^\alpha \Theta + (S_1^\alpha,\, S_2^\alpha,\, S_3^\alpha) + \partial^\alpha \textbf{Src} \tag{7.24}$$
$$(S_1^\alpha,\, S_2^\alpha,\, S_3^\alpha) \overset{\text{def}}{=} \partial^\alpha\big((H - \not{H})\Theta\big) + [\partial^\alpha, \not{H}]\Theta + [\mathbf{M}^\mu - \not{\mathbf{M}}^\mu, \partial^\alpha]\partial_\mu \Theta.$$

If $R \geq 4$, (7.20), (7.21) and **(E9)** imply that

$$E^0\{S_i^\alpha\} \lesssim_R \Big\{ \sum_{j=1}^3 E^R\{H_{ij} - \not{H}_{ij}\} + \frac{|\not{H}_2|^2 + |\not{H}_3|^2}{|t|^4} + \sum_{\mu=0}^3 E^R\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\} \Big\}\, E^R\{\Theta\}.$$

If $R \geq 0$, (7.22), (7.23) and **(E9)** imply that

$$E^0\{S_i^\alpha\} \lesssim_R \Big\{ \sum_{j=1}^3 \big(\textbf{Sup}^{(R)}\{H_{ij} - \not{H}_{ij}\}\big)^2 + \frac{|\not{H}_2|^2 + |\not{H}_3|^2}{|t|^4} + \sum_{\mu=0}^3 \big(\textbf{Sup}^{(R)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}\big)^2 \Big\}\, E^R\{\Theta\}.$$

If **(E11a)** or **(E11b)**, it follows from the inequalities just above that

$$\left.\begin{aligned} |t|^2\, E^0\{S_1^\alpha\}(t) \\ E^0\{S_2^\alpha\}(t) \\ |t|^2\, E^0\{S_3^\alpha\}(t) \end{aligned}\right\} \lesssim_R \ \mathbf{c}_*^2\, E^R\{\Theta\}(t) \tag{7.25}$$

where $\mathbf{c}_* = \max\left\{\mathbf{c}_2,\ |t^*|^{-1}(|\not{H}_2|^2 + |\not{H}_3|^2)^{1/2}\right\}$.

Estimates: We derive the energy inequalities 7.33, stated below. For $i = 1, 2, 3$, define "energy currents" associated to $\Theta = (\Theta_1,\, \Theta_2,\, \Theta_3)$ (see **(E4)**)

$$j_i^\mu[\Theta_i](q) = \left(\Theta_i^T\, M_i^\mu\, \Theta_i\right)(q). \tag{7.26}$$

(Warning: we never sum over repeated "lower" indices.) The important current identity

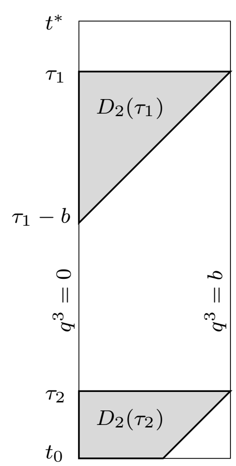
$$\partial_\mu j_i^\mu[\Theta_i] = \Theta_i^T\,(\partial_\mu M_i^\mu)\,\Theta_i + 2\,\Theta_i^T\left(M_i^\mu\,\partial_\mu\Theta_i\right), \tag{7.27}$$

follows from $(M_i^\mu)^T = M_i^\mu$, see **(E1)**. For $\tau \in \overline{\mathcal{I}}$, let $\tau_- = \max\{t_0,\, \tau - 2\}$. Let

$$\mathcal{D}_1(\tau) = \mathcal{D}_3(\tau) = \left\{(t, \mathbf{q}) \in \mathcal{U} \;\middle|\; t \in (\tau_-, \tau)\right\} \tag{7.28}$$

$$\mathcal{D}_2(\tau) = \mathcal{D}_1(\tau) \cap \{\, q \mid q^3 - q^0 < b - \tau\}. \tag{7.29}$$

For the case $i = 2$, see the nearby figure, where $t_0 < \tau_2 < t_0 + b < \tau_1 < t^*$. Energy estimates are obtained by integrating (7.27) over $\mathcal{D}_i(\tau) \subset \mathcal{U} = \mathcal{I} \times \mathcal{O}$ and applying the Euclidean divergence theorem. The divergence theorem generates integrals over the boundary $\partial\mathcal{D}_i(\tau)$, which we now discuss. Recall **(E6)**. The $q^0 = \tau$ boundary contributes $\mathbf{E}_i^0\{\Theta_i\}(\tau)$. There is no contribution from the $q^3 = 0$, by **(E6)**. For $i = 1, 3$, the $q^0 = \tau_-$ boundary contributes $-\mathbf{E}_i^0\{\Theta_i\}(\tau_-)$, and the contribution from $q^3 = b$ is non-negative, by **(E1)**. If $i = 2$, there is always a boundary contribution from $q^3 - q^0 = b - \tau$ and it vanishes by **(E7)**. If $i = 2$ and $\tau < t_0 + b$, there is an additional boundary contribution at $q^0 = t_0$ which is $\geq -\mathbf{E}_2^0\{\Theta_2\}(t_0)$.



The discussion of the last paragraph literally transposes from Θ_i and $j_i^\mu[\Theta_i]$ to $\partial^\alpha\Theta_i$ and $j_i^\mu[\partial^\alpha\Theta_i]$, for $|\alpha| \leq R$. The current $j_i^\mu[\partial^\alpha\Theta_i]$ is C^1 and extends, with its derivatives, continuously to $\overline{\mathcal{U}}$. The preceding analysis of the boundary terms gives the general inequalities

$$\mathbf{E}_i^0\{\partial^\alpha\Theta_i\}(\tau) - k_i(\tau)\,\mathbf{E}_i^0\{\partial^\alpha\Theta_i\}(\tau_-) \leq \int_{\mathcal{D}_i(\tau)} \mathrm{d}^4 q\ \ \partial_\mu j_i^\mu[\partial^\alpha\Theta_i](q). \tag{7.30}$$

for $i = 1, 2, 3$. Here, by definition, $k_1(\tau) = k_3(\tau) = 1$ for all τ, whereas $k_2(\tau)$ vanishes when $\tau_- > t_0$ and is equal to 1 when $\tau_- = t_0$. Summing over $|\alpha| \leq R$,

$$\mathbf{E}_i^R\{\Theta_i\}(\tau) - k_i(\tau)\,\mathbf{E}_i^R\{\Theta_i\}(\tau_-) \leq \int_{\mathcal{D}_i(\tau)} \mathrm{d}^4 q \sum_{|\alpha| \leq R} \partial_\mu j_i^\mu[\partial^\alpha\Theta_i](q). \tag{7.31}$$

The current identity (7.27), with $\partial^\alpha \Theta_i$ in the role of Θ_i, and (7.24) imply

$$\begin{aligned}
&\partial_\mu j_i^\mu[\partial^\alpha \Theta_i](q) \qquad (7.32)\\
&= 2(\partial^\alpha\Theta_i)^T \Big\{ \Big(\sum_{j=1}^{3} \not{H}_{ij}\,(\partial^\alpha \Theta_j)\Big) + \tfrac{1}{2}\,(\partial_\mu M_i^\mu)(\partial^\alpha\Theta_i) + (\partial^\alpha \mathbf{Src}_i) + S_i^\alpha \Big\}.
\end{aligned}$$

For $i = 1, 2$, we directly estimate the right hand side of (7.31), by using Schwarz's inequality for the spatial part of the integral, and **(E6)**, **(E10)**, (7.25) and (7.18). For $i = 3$, we first exploit $\not{H}_3 \leq 0$ (see **(E9)**) to drop the term $2(\partial^\alpha\Theta_3)^T \not{H}_{33}(\partial^\alpha\Theta_3)$, and then go on as before. We also use the estimate $|\partial_\mu \mathbf{M}^\mu| = |\partial_\mu(\mathbf{M}^\mu - \not{\mathbf{M}}^\mu)| \lesssim_R \mathbf{c}_2\,|t|^{-1} \leq \mathbf{c}_*\,|t|^{-1}$ that holds when either **(E11a)** or **(E11b)** is assumed (in the first case, we use (7.19)). Abbreviating $\mathbf{E}_i = \mathbf{E}_i^R\{\Theta_i\}$ and $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$, we have for all $\tau \in \overline{\mathcal{I}}$:

$$\begin{aligned}
\mathbf{E}_1(\tau) - \mathbf{E}_1(\tau_-) &\lesssim_X \int_{\tau_-}^{\tau} \frac{\mathrm{d}t}{|t|}\,\sqrt{\mathbf{E}_1(t)}\Big(\mathbf{c}_*\sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J}\Big)\\
\mathbf{E}_2(\tau) - \mathbf{E}_2(t_0) &\lesssim_X \int_{\tau_-}^{\tau} \mathrm{d}t\,\sqrt{\mathbf{E}_2(t)}\Big(\sqrt{\mathbf{E}_1(t)} + \mathbf{c}_*\sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J}\Big) \qquad (7.33)\\
\mathbf{E}_3(\tau) - \mathbf{E}_3(\tau_-) &\lesssim_X \int_{\tau_-}^{\tau} \frac{\mathrm{d}t}{|t|}\,\sqrt{\mathbf{E}_3(t)}\Big(\sqrt{\mathbf{E}_2(t)} + \mathbf{c}_*\sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J}\Big)
\end{aligned}$$

where X is defined as in the proposition.

For each $A = (A_1, A_2, A_3) \in (0, \infty)^3$, define

$$\mathcal{J}(A) = \Big\{ t \in \overline{\mathcal{I}} \;\Big|\; \sup_{\tau \in [t_0, t]} |\tau|^{2J}\mathbf{E}_i(\tau) \leq A_i^2, \quad i = 1, 2, 3 \Big\}$$

Assume A satisfies (recall that $J \geq J_0 > 0$, by assumption)

$$\begin{aligned}
&A_1 > |t_0|^J\sqrt{\mathbf{E}_1(t_0)} && A_1 > \tfrac{C}{J_0}\,\mathbf{c}_1 && A_1 > \tfrac{C}{J_0}\mathbf{c}_*|A|\\
&A_2 > 2|t_0|^J\sqrt{\mathbf{E}_2(t_0)} && A_2 > 8C\mathbf{c}_1 && A_2 > 8C(A_1 + \mathbf{c}_*|A|) \qquad (7.34)\\
&A_3 > |t_0|^J\sqrt{\mathbf{E}_3(t_0)} && A_3 > \tfrac{C}{J_0}\mathbf{c}_1 && A_3 > \tfrac{C}{J_0}(A_2 + \mathbf{c}_*|A|)
\end{aligned}$$

where $|A|^2 = A_1^2 + A_2^2 + A_3^2$ and where $C = C(X) > 0$ is the maximum of the three constants of proportionality in the inequalities (7.33). It is a direct consequence of the inequalities (7.33), (7.34) and the continuity of $\overline{\mathcal{I}} \ni \tau \mapsto \mathbf{E}_i(\tau)$ that $\mathcal{J}(A)$ is an open and closed sub-interval of $\overline{\mathcal{I}}$ which contains t_0. Therefore, $\mathcal{J}(A) = \overline{\mathcal{I}}$. To see that $\mathcal{J}(A)$ is open in $\overline{\mathcal{I}}$, first observe that for every $\tau \in \mathcal{J}(A)$, the inequalities (7.33), (7.34) imply the strict inequalities $\mathbf{E}_i(\tau) < (A_i|\tau|^{-J})^2$, and then use continuity.

For each $\lambda \geq 0$, set

$$A(\lambda) = \lambda\Big(1\ ,\ 1 + 8C\ ,\ 1 + \tfrac{C}{J_0}(1 + 8C)\Big).$$

The three rightmost inequalities in (7.34) are homogeneous (degree 1) in A, and hold for $A(\lambda)$, $\lambda > 0$, if and only if they hold for $A(1)$, which is the case if $\mathbf{c}_* > 0$ is sufficiently small depending only on X, because it is true for $\mathbf{c}_* = 0$. The definition

of $\mathbf{c}_*$ right after (7.25) implies $\mathbf{c}_* \le (1 + |\not{H}_2|^2 + |\not{H}_3|^2)^{1/2}\, \mathbf{c}_3(X)$. Consequently, the condition on $\mathbf{c}_*$ holds if $\mathbf{c}_3(X)$ is suitably small. If

$$\lambda \,>\, \lambda_0 \overset{\text{def}}{=} 2|t_0|^J \sqrt{\mathbf{E}(t_0)} + \max\{8,\, J_0^{-1}\}\, C\, \mathbf{c}_1 \,\ge\, 0$$

then the remaining inequalities in (7.34) hold for $A(\lambda)$, that is $\mathcal{J}(A(\lambda)) = \overline{\mathcal{I}}$. By the definition of $\mathcal{J}(A)$, we have $\mathcal{J}(A(\lambda_0)) = \overline{\mathcal{I}}$. By (7.18), inequality (7.16) follows if $\mathbf{c}_4(X)$ is sufficiently big. □

Remark 7.1. Once the system of inequalities (7.33) has been established, the rest of the argument is abstract, in the sense that it holds for any three functions $\mathbf{E}_i$, $i = 1, 2, 3$, satisfying (7.33).

7.5. Refined energy estimate. We proved a finite speed of propagation theorem for formal power series vacuum fields $[\varPsi]$, see Proposition 6.2. The refined energy estimate obtained in this subsection plays a similar role for a classical vacuum field $\varPsi$.
To make the last statement more precise, recall that the energy estimate for the symmetric hyperbolic system (7.12) was obtained by integrating the divergence current identity (7.27) over appropriate open subsets $\mathcal{C} \subset \mathbb{R}^4$. We now construct more refined sets $\mathcal{C}$ which allow us to estimate the energy "localized in the (ξ^1, ξ^2) plane". We will be guided by the basic requirement that the boundary integrals in the divergence theorem have definite signs. That is, the boundary of $\mathcal{C}$ must be non-timelike (with respect to the symmetric hyperbolic system).

Convention 7.2. Until further notice, we use the coordinates

$$x = (x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u).$$

Recall the matrix differential operators $\mathbf{A}(\varPhi)$ and $\widehat{\mathbf{A}}(\varPhi)$ associated to $\varPhi = (e, \gamma, w)$. See, (2.5), (2.8). Suppose θ is a one-form and suppose (see, (2.2))

$$\theta_\mu L^\mu \ge 0, \qquad \theta_\mu N^\mu \ge 0, \qquad \theta_\mu \begin{pmatrix} N & D \\ \overline{D} & L \end{pmatrix}^\mu \ge 0 \tag{7.35}$$

The last inequality is in the sense of Hermitian matrices. Then

$$\theta_\mu \mathbf{A}^\mu(\varPhi) \ge 0, \qquad \theta_\mu \widehat{\mathbf{A}}^\mu(\varPhi) \ge 0. \tag{7.36}$$

For each $x_0 = (\xi_0, \underline{u}_0, u_0) \in \mathbb{R}^2 \times (0, \infty) \times (-\infty, 0)$ and choice of constants $k_0, k_1, \mathfrak{d} > 0$, where $\mathfrak{d} < \underline{u}_0$ and $\mathfrak{d} < |u_0|^{-1}$, set

$$\mathcal{C} = \bigcup_{(\underline{u},u)\,\in\,\mathcal{B}} \Big(D_{r(\underline{u},u)}(\xi_0) \times \{(\underline{u}, u)\} \Big)$$
$$\mathcal{F} = \bigcup_{(\underline{u},u)\,\in\,\mathcal{B}} \Big(\partial D_{r(\underline{u},u)}(\xi_0) \times \{(\underline{u}, u)\} \Big)$$

where

$$\mathcal{B} = (0, \underline{u}_0 - \mathfrak{d}) \times \big(-\infty, -\tfrac{1}{|u_0|^{-1} - \mathfrak{d}} \big) \subset \mathbb{R}^2,$$
$$r(\underline{u}, u) = k_0 + k_1\, |\underline{u}_0 - \underline{u}| \cdot \big| |u_0|^{-1} - |u|^{-1} \big|. \tag{7.37}$$

More geometrically, $\mathcal{C}$ is a disk bundle over the base $\mathcal{B}$, and $\mathcal{F}$ the corresponding circle bundle. The set $\mathcal{C}$ is an open subset of $\mathbb{R}^4$. Note that, $r : \mathcal{B} \to (k_0,\ k_0 + k_1|\underline{u}_0|/|u_0|)$, a bounded set. The set $\mathcal{C}$ has piecewise smooth boundary. We concentrate on the smooth piece $\mathcal{F} \subset \partial\mathcal{C}$ here. Let θ be a 1-form along $\mathcal{F}$ whose kernel coincides with the tangent space to $\mathcal{F}$ and for which $\theta(X) > 0$ if X is a vector pointing out of $\mathcal{C}$. We choose

$$\theta = \sum_{i=1}^{2} \widehat{\xi}^i \, \mathrm{d}\xi^i + k_1 \left| \frac{1}{|u|} - \frac{1}{|u_0|} \right| \mathrm{d}\underline{u} + k_1 \frac{|\underline{u}_0 - \underline{u}|}{|u|^2} \, \mathrm{d}u, \qquad \widehat{\xi}^i = \frac{\xi^i - \xi_0^i}{|\xi - \xi_0|}. \tag{7.38}$$

Proposition 7.5. *Let $x \in \mathcal{F}$. If $e_3(x) > 0$ and the inequality*

$$k_1 \mathfrak{d} \geq \ 2 \max \left\{ \frac{|u|}{\sqrt{e_3}} \sqrt{|e_1|^2 + |e_2|^2},\ \ |u|^2 \sqrt{|e_4|^2 + |e_5|^2} \right\} \tag{7.39}$$

holds at x, then (7.36) *holds at x with θ given by* (7.38)*.*

Proof. If $\theta_\mu L^\mu > 0$ and $\theta_\mu N^\mu > 0$ and $\det \theta_\mu \left(\begin{smallmatrix} N & D \\ D & L \end{smallmatrix} \right)^\mu > 0$, then (7.35) and therefore (7.36) hold. The condition $e_3 > 0$ implies $\theta_\mu L^\mu > 0$. By (7.39),

$$\widehat{\xi}^1 e_4 + \widehat{\xi}^2 e_5 + k_1 \, |u|^{-2} \, |\underline{u}_0 - \underline{u}| \ \geq \ \frac{k_1 \mathfrak{d}}{2|u|^2} > 0$$

which implies $\theta_\mu N^\mu > 0$. Finally, $e_3 > 0$ and (7.39) imply

$$e_3 \, k_1 \left| |u|^{-1} - |u_0|^{-1} \right| \left(\widehat{\xi}^1 e_4 + \widehat{\xi}^2 e_5 + k_1 \, |u|^{-2} \, |\underline{u}_0 - \underline{u}| \right) - \left| \widehat{\xi}^1 e_1 + \widehat{\xi}^2 e_2 \right|^2 > 0$$

and therefore $\det \theta_\mu \left(\begin{smallmatrix} N & D \\ D & L \end{smallmatrix} \right)^\mu > 0$. $\square$

Remark 7.2. Proposition 7.5 will be applied as follows. Fix Φ, and consider symmetric hyperbolic systems with differential operators given by $\mathbf{A}(\Phi)$ or $\widehat{\mathbf{A}}(\Phi)$. Then, if the assumptions of Proposition 7.5 are satisfied for all points $x \in \mathcal{F}' \subset \mathcal{F}$, the boundary integral $\int_{\mathcal{F}'} \langle j, \nu \rangle$, where j is the energy current vector field, is non-negative.

Convention 7.3. Observe that the definitions of $\mathcal{C}$ and $\mathcal{F}$ depend only on the parameters k_0, k_1, $\mathfrak{d}$, ξ_0, $\underline{u}_0$, u_0. **For the rest of this paper**, we make the specific choice of parameters

$$\mathfrak{d} = 10^{-3}, \qquad k_0 = \tfrac{1}{4}, \qquad k_1 = \tfrac{1}{18}\mathfrak{d}^{-1}, \qquad u_0 = -\tfrac{1}{2}\mathfrak{d}^{-1}, \qquad \underline{u}_0 = b + \mathfrak{d} \tag{7.40}$$

For each $b \in [1, 2]$ and $\xi_0 \in \mathbb{R}^2$, we denote the corresponding sets by $\mathcal{C}(\xi_0, b)$ and $\mathcal{F}(\xi_0, b)$. The base $\mathcal{B}$ is given by $\mathcal{B} = (0, b) \times (-\infty, -\mathfrak{d}^{-1})$ and the radius function $r(\underline{u}, u)$ takes values in $(\frac{1}{4}, \frac{1}{2})$ on $\mathcal{B}$.

Recall the far field ansatz (see, Section 5) $\Phi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\,\Psi$, where $\Psi = (f, \omega, z)$. The ansatz depends on the scaling parameters a and $\mathfrak{A}$.

Convention 7.4. **For the rest of this paper**, the parameters $a, \mathfrak{A} \in \mathbb{R}$ are restricted by

$$0 < |\mathfrak{A}| \ \leq \ |a| \ \leq \ 10^{-3} \tag{7.41}$$

Proposition 7.6. *Let* $\mathfrak{d}$ *be fixed as in* (7.40)*. Let* $\xi_0 \in \mathbb{R}^2$ *and* $b \in [1,2]$*. Assume* $a, \mathfrak{A} \in \mathbb{R}$ *satisfy* (7.41)*. If, in addition,*

$$|\Psi(x)| \ \leq \ 5$$

at $x = (\xi, \underline{u}, u) \in \mathcal{F}(\xi_0, b)$ *and* $|\xi| < 4\,|\frac{a}{\mathfrak{A}}|$*, then the assumptions of Proposition 7.5 hold at* x*.*

Proof. The first five components of Ψ satisfy $|f_i| \leq |\Psi| \leq 5$. Consequently, the first five components of Φ satisfy

$$\begin{aligned} |u|\,|e_i| &\leq |u|\big|\rho_{a,\mathfrak{A}}^{-1}\mathbf{e}_{a,\mathfrak{A}}\big| + |u|\,\big|\tfrac{1}{u^2}f_i\big| \leq |\mathbf{e}_{a,\mathfrak{A}}| + \tfrac{1}{|u|}|f_i| \leq \tfrac{17}{2}\,|a| + 5\mathfrak{d} \qquad i = 1,2 \\ e_3 &= 1 + \tfrac{1}{u^2}f_3 \geq 1 - \tfrac{1}{|u|^2}|f_3| \geq 1 - 5\mathfrak{d}^2 \geq (\tfrac{3}{4})^2 \\ |u|^2\,|e_i| &= \tfrac{1}{|u|}|f_i| \leq \tfrac{5}{|u|} \leq 5\mathfrak{d} \qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 4,5 \end{aligned}$$

In the first line, we use $|\xi| < 4\,|\frac{a}{\mathfrak{A}}|$. The proposition follows by direct inspection. $\square$

Convention 7.5. For the rest of this subsection, we use coordinates

$$q = (q^0,\, q^1,\, q^2,\, q^3) = \big(t = u + \underline{u}\,,\ \xi^1,\, \xi^2,\, \underline{u}\big)$$

Assumptions for the refined energy estimate. $\mathfrak{d}$ is defined in (7.40).

(RE0) $\mathcal{I} = (t_0, t^*)$ where $-\infty < t_0 < t^* < -\mathfrak{d}^{-1}$, $\ \xi_0 \in \mathbb{R}^2$, $\ b \in [1,2]$,

$$\begin{aligned} \mathcal{U} &= \bigcup_{t \in \mathcal{I}} \Big(\{t\} \times \mathcal{O}(\xi_0, b, t)\Big) \ \subset \ \mathbb{R}^4 \\ \mathcal{O}(\xi_0, b, t) &= \bigcup_{\underline{u} \in (0,b)} \Big(D_{r'(t,\underline{u})}(\xi_0) \times \{\underline{u}\}\Big), \\ r'(t, \underline{u}) &= \tfrac{1}{4} + \tfrac{1}{18\mathfrak{d}}|b + \mathfrak{d} - \underline{u}| \cdot \big|2\mathfrak{d} - \tfrac{1}{\underline{u} + |t|}\big|. \end{aligned}$$

(RE1) - (RE9) are formulated identically to **(E1) - (E9)**.

(RE10), (RE11a), (RE11b) are formulated identically to **(E10), (E11a), (E11b)** with the understanding that $E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$ are replaced by $E^R_{\mathcal{O}(\xi_0,b,t)}$ and $\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}$, see (7.14) and (7.15).

(RE12) Let the 1-form θ be as in (7.38). Then, $\theta_\mu\, \mathbf{M}^\mu \geq 0$ on

$$(\partial\mathcal{U}) \cap \big(\mathcal{I} \times \mathbb{R}^2 \times (0,b)\big)$$

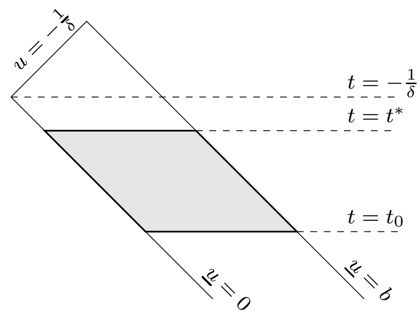
Remark 7.3. $\mathcal{U}$ is a bundle over $\mathcal{I}$ with fiber $\mathcal{O}(\xi_0, b, t) \subset \mathbb{R}^3$ at $t \in \mathcal{I}$. The fiber is an open disk bundle over the $\underline{u}$-interval $(0,b)$. An equivalent description of the fiber is

$$\mathcal{O}(\xi_0, b, t) = \Big\{\mathbf{q} = (\xi^1,\, \xi^2,\, \underline{u}) \in \mathbb{R}^3 \ \Big| \ (\xi^1,\, \xi^2,\, \underline{u},\, t - \underline{u}) \in \mathcal{C}(\xi_0, b)\Big\}$$

for each $t \in \mathcal{I}$. It is important that

$$(\partial\mathcal{U}) \cap \big(\mathcal{I} \times \mathbb{R}^2 \times (0,b)\big) \ \ \subset \ \ \mathcal{F}(\xi_0, b).$$

For each $t \in \mathcal{I}$, the map $r'(t,\,\cdot\,) :\ (0,b) \to (\frac{1}{4}, \frac{1}{2})$ is decreasing.



Proposition 7.7 (Refined Energy Estimate). *Suppose that the refined hypotheses* **(RE0)** *through* **(RE10)** *and* **(RE12)** *hold, and, also, either* **(RE11a)** *or* **(RE11b)** *holds. Let $J_0 > 0$ and assume $J \geq J_0$, see* **(RE10)***. Then, there are constants $\mathbf{c}_3(X) \in (0,1)$, $\mathbf{c}_4(X) > 0$ depending only on $X = \big(R, J_0, |\not{H}_1|, |\not{H}_2|, |\not{H}_3|\big)$, such that $\mathbf{c}_2 \leq \mathbf{c}_3(X)$ and $|t^*|^{-1} \leq \mathbf{c}_3(X)$ imply that*

$$\sqrt{E^R_{\mathcal{O}(\xi_0,b,\tau)}\{\Theta\}(\tau)} \leq \mathbf{c}_4(X)\, \frac{|t_0|^J \sqrt{E^R_{\mathcal{O}(\xi_0,b,t_0)}\{\Theta\}(t_0)} + \mathbf{c}_1}{|\tau|^J}. \tag{7.42}$$

for all $\tau \in \mathcal{I}$ (see, **(E0)** *for the definition of $\mathcal{I}$).*

Remark 7.4. We use the same names for the constants in the assumptions and statements of both energy estimates, Propositions 7.4 and 7.7. This was done for convenience, and does not imply that there is any relationship between them.

Proof. This proof completely mimics the proof of Proposition 7.4 with a few modifications. First of all, our previous conventions that $E^R = E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$ are replaced by the conventions $E^R = E^R_{\mathcal{O}(\xi_0,b,t)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}$. Also, the definitions (7.17) are replaced by

$$\mathbf{E}^0_i\{f\}(t) = \int_{\mathcal{O}(\xi_0,b,t)} \mathrm{d}^3\mathbf{q}\, \big(f^T M^0_i f\big)(t,\mathbf{q}) \quad , \quad \mathbf{E}^R_i\{f\}(t) = \sum_{\substack{|\alpha| \leq R \\ \alpha \in \mathbb{N}^4_0}} \mathbf{E}^0_i\{\partial^\alpha f\}(t)$$

The inequalities (7.18), (7.19), (7.20), (7.21), (7.22), (7.23) still hold with these modifications. The only one that requires discussion is the Sobolev inequality (7.19). For this purpose, let $\mathbf{CYL} = D_{\frac{1}{4}}(0) \times (0,b)$ and $\phi :\ \mathbf{CYL} \to \mathcal{O}(\xi_0,b,t)$ be the diffeomorphism $\phi(\xi,\underline{u}) = \big(\xi_0 + 4\, r'(t,\underline{u})\,\xi,\ \underline{u}\big)$. Then,

$$\begin{aligned} \mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) &\lesssim_R \mathbf{Sup}^{(R-2)}_{\mathbf{CYL}}\{f\circ\phi\}(t) \\ &\lesssim_R \sqrt{E^R_{\mathbf{CYL}}\{f\circ\phi\}(t)} \lesssim_R \sqrt{E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)}. \end{aligned} \tag{7.43}$$

The second inequality follows from Lemma 7.2. The first and third inequalities are direct consequences of the chain rule, because all derivatives of order up to $R-1$ of the Jacobians of ϕ and ϕ^{-1} have finite sup-norms on their domains of definition depending *only* on R, especially, independent of ξ_0, b and t.

Observe that in (7.28), (7.29), the set $\mathcal{U}$ is now given as in **(RE0)**. Estimate (7.30) still holds. By construction, $D_1(\tau)$ is a disk bundle over the $(t,\underline{u})$-**rectangle** $(\tau,\tau_-) \times (0,b)$.

The boundary $\partial D_1(\tau)$ has five components, four of them arising as disk bundles over the boundary of the **rectangle**, the fifth is a circle bundle over the interior. The treatment of the first four components is unchanged. The fifth is accounted for by **(RE12)**. The domains $D_2(\tau)$ and $D_3(\tau)$ are handled in the same way.

The rest of the proof is completely unchanged. $\square$

8. Classical Vacuum Fields

In Section 6 we have constructed formal power series vacuum fields $[\,\Psi\,]$ by solving an initial value problem for (5.7a). The goal of this section is to prove the existence of an actual, classical, vacuum field Ψ, for which $[\,\Psi\,]$ is rigorously an asymptotic expansion.

Convention 8.1. We adopt the conventions of Section 7. We use the coordinates $q = (t, \mathbf{q})$ (see, Convention 7.1). Keep in mind that $u = u(q) = q^0 - q^3$. To conveniently translate between the x coordinate system (Sections 2 through 6) and the q coordinate system (Sections 7 and 8), we abuse notation and write $f(q)$ instead of $f(x(q))$, for any function f. It is also implicit that partial derivatives are adapted to the new coordinate system. For example, the matrix differential operator $\mathbf{A}^\mu(x(q), \Psi)\frac{\partial}{\partial x^\mu}$ is abbreviated as $\mathbf{A}^\mu(q, \Psi)\frac{\partial}{\partial q^\mu}$.

8.1. *Preparatory Definitions and Estimates.*

The goal of this subsection is to make the necessary definitions and estimates so that the Existence/Breakdown Theorem and the Refined Energy Estimate can be applied to (5.7a) and (5.7b).

Convention 8.2. (5.7a) and (5.7b) are equivalent to *real* symmetric hyperbolic systems for $\mathcal{R} \cong \mathbb{R}^{31}$ and $\widehat{\mathcal{R}} \cong \mathbb{R}^{32}$ valued fields, respectively. See, Remark 2.8 and Proposition 5.1. This equivalence will be implicit each time the Existence/Breakdown theorem and the Refined Energy Estimate are applied to (5.7a) and (5.7b), or to equivalent systems.

Convention 8.3. In this section, $\mathbb{C}^m$ is a vector space over $\mathbb{R}$ with dimension $2m$. A linear map from $\mathbb{C}^m$ to $\mathbb{C}^n$ is, by convention, linear over $\mathbb{R}$. It can be represented either as a $2n \times 2m$ real matrix, or as an $n \times m$ complex matrix which may have the complex conjugation operator C as matrix elements. We adopt similar conventions for the real subspaces $\mathcal{R} \subset \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5$ and $\widehat{\mathcal{R}} \subset \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3$.

Convention 8.4. The notation $F(q, f, \partial_q f, \ldots)$ displays the explicit pointwise dependence of F on q, $f(q)$, $\partial_q f(q)$, $\ldots$

To put (5.7a) in the form required by Propositions 7.3 and 7.7, we use

(S1) $a, \mathfrak{A} \in \mathbb{R}$ satisfy Convention 7.4.

(S2) $[\,\Psi\,] = \sum_{k=0}^{\infty} (\frac{1}{u})^k\, \Psi(k)$ is the formal power series solution in Proposition 6.2 corresponding to **DATA**$(\xi, \underline{u}) =$ **DATA**$(\mathbf{q})$ which vanishes for $q^3 < \underline{u}_0 = \frac{1}{2}$. Therefore, $[\,\Psi\,]$ vanishes when $q^3 < \frac{1}{2}$ by Proposition 6.2. Fix an integer $K \geq 0$, and set $\Psi_K = \sum_{k=0}^{K+1} (\frac{1}{u})^k\, \Psi(k)$.

(S3) The field Ψ is expressed in terms of (h,σ,ℓ) by

$$\Psi \;=\; (f,\omega,z) \;=\; \Psi_K + (h,\sigma,\ell). \tag{8.1}$$

Let $\Xi = (\Xi_1,\Xi_2,\Xi_3)$ be the field given by

$$\begin{aligned}&(\Xi_1,\ \Xi_2,\ \Xi_3)\\ &= \big(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5\big) \oplus \big(h_1,h_2,h_4,h_5,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_8\big) \oplus \big(h_3,\sigma_5,\sigma_6,\sigma_7\big)\end{aligned}$$

There is a permutation matrix π so that

$$(h,\sigma,\ell) \;=\; \pi\,(\Xi_1,\Xi_2,\Xi_3) \tag{8.2}$$

The field $\Xi = (\Xi_1,\Xi_2,\Xi_3)$ takes values in $\pi^{-1}\mathcal{R} \,\subset\, \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$. The permutation is required for Proposition 7.7, see **(S6)**.

(S4) System (5.7a) is abbreviated as $\mathbf{A}(q,\Psi)\,\Psi = \mathbf{f}(q,\Psi)$ (see, Convention 8.1). Some of its properties are discussed in Remark 5.3. System (5.7a) is equivalent to

$$\mathbf{B}(q,\Xi)\Xi \;=\; Q(q,\Xi)\,\Xi + \textbf{Src}(q) \quad , \quad \mathbf{B} = \mathbf{B}^\mu \tfrac{\partial}{\partial q^\mu} \tag{8.3a}$$

$$\mathbf{B}(q,\Xi) = \pi^{-1}\mathbf{A}(q,\Psi_K + \pi\,\Xi)\,\pi \tag{8.3b}$$

$$Q(q,\Xi)\Pi = \pi^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_0^1 \mathrm{d}s'\Big(- \mathbf{A}(q,s\pi\Pi)\Psi_K + \mathbf{f}(q,\Psi_K + s'\pi\Xi + s\pi\Pi)\Big) \tag{8.3c}$$

with the source term

$$\textbf{Src}(q) = \pi^{-1}\,\big(\mathbf{f}(q,\Psi_K) - \mathbf{A}(q,\Psi_K)\Psi_K\big).$$

The transformation $Q(q,\Xi)$ acting on $\pi^{-1}\mathcal{R}$ is linear over $\mathbb{R}$. Note that the bracketed expression in (8.3c) is a quadratic polynomial in s and s'. The operator $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\int_0^1 \mathrm{d}s'$ selects certain combinations of its coefficients.

(S5) The matrices $\mathbf{B}^\mu$ and Q are affine linear (over $\mathbb{R}$) in Ξ. Let $\overset{\bullet}{\mathbf{B}}{}^\mu(q)$ and $\overset{\bullet}{Q}(q)$ be the $\mathbb{R}$ linear maps given by

$$\overset{\bullet}{\mathbf{B}}{}^\mu(q)\Pi = \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,\mathbf{B}^\mu(q,s\Pi) \quad , \quad \overset{\bullet}{Q}(q)\Pi = \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,Q(q,s\Pi)$$

We have, $\mathbf{B}^\mu(q,\Xi) = \mathbf{B}^\mu(q,0) + \overset{\bullet}{\mathbf{B}}{}^\mu(q)\,\Xi$. Similarly for Q.

(S6) The three by three $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block-decomposition of $\mathbf{B}$ is

$$\mathbf{B} = \mathrm{diag}(B_1,B_2,B_3), \qquad B_2 = \mathbb{1}_9\,L, \qquad B_3 = \mathbb{1}_4\,N,$$

and B_1 is the 5×5 Hermitian matrix operator on the left hand side of (5.10c). The block-decomposition of Q is denoted $Q = (Q_{mn})_{m,n=1,2,3}$.

(S7) $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ are constant 9×5, 4×9, 4×4 matrices. Their nonzero entries are (C is the complex conjugation operator):

$$\begin{array}{lll}(\mathcal{Q}_1)_{51} = -1 & (\mathcal{Q}_1)_{72} = -1 & (\mathcal{Q}_1)_{93} = 1\\ (\mathcal{Q}_2)_{19} = -1 - C & (\mathcal{Q}_2)_{27} = C & (\mathcal{Q}_2)_{28} = -1\\ (\mathcal{Q}_3)_{11} = -2 & (\mathcal{Q}_3)_{22} = -1 & \end{array}$$

Observe that $\mathcal{Q}_3 \le 0$. Let $\mathcal{Q}(t) = (\mathcal{Q}_{mn})_{m,n=1,2,3}$ be the $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block matrix

$$\mathcal{Q}(t) = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{Q}_1 & 0 & 0 \\ 0 & |t|^{-1}\mathcal{Q}_2 & |t|^{-1}\mathcal{Q}_3 \end{pmatrix}$$

(S8) $\not{B}^\mu$ are the constant, diagonal, $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block matrices

$$\not{\mathbf{B}} = \operatorname{diag}(U,U,U,U,V) \,\oplus\, \mathbb{1}_9\, V \,\oplus\, \mathbb{1}_4\, U \quad , \quad \not{\mathbf{B}} = \not{\mathbf{B}}^\mu \tfrac{\partial}{\partial q^\mu}$$

where $U = \frac{\partial}{\partial q^0}$ and $V = \frac{\partial}{\partial q^0} + \frac{\partial}{\partial q^3}$. Note that $\not{\mathbf{B}}^0 = \mathbb{1}_{18}$, $\not{\mathbf{B}}^1 = \not{\mathbf{B}}^2 = 0$, $\not{\mathbf{B}}^3 \ge 0$.

(S9) Let $s(x)$ be the smooth function that vanishes when $x \le 0$ and is equal to $e^{-1/x}$ when $x > 0$. Let $\psi = \psi(\mathbf{q}) :\ \mathbb{R}^3 \to [0,1]$ be the smooth cutoff function

$$\psi(\mathbf{q}) = \frac{s\big(3 - |\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2}\big)}{s\big(3 - |\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2}\big) + s\big(|\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2} - \frac{5}{2}\big)}\ \frac{s\big(\frac{3}{4} - |q^3 - 1|\big)}{s\big(\frac{3}{4} - |q^3 - 1|\big) + s\big(|q^3 - 1| - \frac{2}{3}\big)}$$

where $\xi = (\xi^1, \xi^2) = (q^1, q^2)$. Let

$$\mathcal{K} = \overline{D_{3|\frac{a}{\mathfrak{A}}|}(0) \times (\tfrac{1}{4}, \tfrac{7}{4})} \qquad \subset \qquad \mathcal{Q} = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0,2)$$

By construction, $\operatorname{supp}_{\mathbb{R}^3} \psi \subset \mathcal{K}$ and ψ is equal to 1 on $D_{\frac{5}{2}|\frac{a}{\mathfrak{A}}|}(0) \times (\frac{1}{3}, \frac{5}{3})$. For each integer $R \ge 0$, the bound $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$ is independent of a and $\mathfrak{A}$, see **(S1)**.

(S10) Define

$$\begin{aligned} \mathbf{M}^\mu(q,\Xi) &= \psi\, \mathbf{B}^\mu(q,\Xi) + (1-\psi)\, \not{\mathbf{B}}^\mu \\ H(q,\Xi) &= \psi\, Q(q,\Xi) + (1-\psi)\, \mathcal{Q}(t) \\ h(q,\Xi) &= H(q,\Xi)\Xi + \psi\, \mathbf{Src}(q) \end{aligned}$$

where $\psi = \psi(\mathbf{q})$ is given in **(S9)**.

(S11) If $\Xi^{(1)}$ and $\Xi^{(2)}$ are both smooth solutions to $\mathbf{B}\,\Xi = Q\,\Xi + \mathbf{Src}$, see **(S4)**, then their difference $\Upsilon = \Xi^{(2)} - \Xi^{(1)}$ is a solution to

$$\begin{aligned} &\mathbf{B}(q,\Xi^{(1)})\Upsilon \;=\; G\,\Upsilon \\ &G\,\Pi \;\overset{\text{def}}{=}\; \tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(Q(q,\Xi^{(1)})(s\Pi) - \mathbf{B}^\mu(q,s\Pi)\,\tfrac{\partial \Xi^{(2)}}{\partial q^\mu} + Q(q,s\Pi)\,\Xi^{(2)}\Big) \end{aligned} \tag{8.4}$$

where $G\big(q,\Xi^{(1)},\Xi^{(2)},\partial_q\Xi^{(2)}\big)$ acts on $\pi^{-1}\mathcal{R}$ linearly over $\mathbb{R}$. The bracketed expression in (8.4) is affine linear in s. The operator $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}$ selects the coefficient of s.

Definition 8.1. *Each entry to the left of the vertical bar is a generic symbol for a polynomial (with complex coefficients) in the (components of the) quantities to the right and their complex conjugates.*

$\mathcal{J}\ \big|\ u^{-1}$

$\boldsymbol{\mathcal{J}}\ \big|\ 1$ *(That is, a generic symbol for a complex number)*

$\mathcal{H}\ \big|$ *linear over* $\mathbb{R}$ *in* $\Psi(0)$

$\boldsymbol{\mathcal{H}}\ \big|$ *linear over* $\mathbb{R}$ *in* $\mathfrak{A},\ \mathbf{e},\ \boldsymbol{\lambda}$

$\mathcal{G}_K\ \big|\ u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(k)_{k=0\ldots K+1},$ *and their first derivatives*

$\boldsymbol{\mathcal{G}}_K\ \big|\ u^{-1},\ \mathfrak{A},\ \underline{u},\ S_K,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(k)_{k=0\ldots K+1},$ *and their first derivatives.*
It has no constant term as a polynomial in $\Psi(k)$ *and its derivatives.*

There is no subscript K on the generic symbols $\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ because they represent polynomials that are required, *by definition, to be independent of K. Precisely, neither their coefficients nor their degrees depend on K. By contrast, the presence of the subscript K on the generic symbols $\mathcal{G}_K$, $\boldsymbol{\mathcal{G}}_K$ indicates that they represent polynomials that are allowed, by definition, to depend on K in an arbitrary manner. Precisely, their coefficients and degrees may be functions of K.*
Above, S_K is defined by $S = -\sum_{k=0}^{K}(\frac{1}{u})^k\,\mathfrak{A}^{2(k+1)}\underline{u}^{k+1} + \frac{1}{u^{K+1}}S_K$, where as before $\frac{1}{\rho} = -\frac{1}{u} + \frac{S}{u^2}$, see (4.4) *and* (5.4).

Proposition 8.1.

$$
\begin{aligned}
\mathbf{B}^\mu(q,0) &= \not{\mathbf{B}}^\mu && + && u^{-1}\boldsymbol{\mathcal{H}} && + && u^{-2}\mathcal{G}_K && \text{(8.5a)}\\
Q_{1n}(q,0) &= \not{Q}_{1n}(q) && + && u^{-1}\mathcal{H} + u^{-1}\boldsymbol{\mathcal{H}} && + && u^{-2}\mathcal{G}_K && \text{(8.5b)}\\
Q_{2n}(q,0) &= \not{Q}_{2n}(q) && + && \mathcal{H} + \boldsymbol{\mathcal{H}} && + && u^{-1}\mathcal{G}_K && \text{(8.5c)}\\
Q_{3n}(q,0) &= \not{Q}_{3n}(q) && + && (|t|^{-1} + u^{-1})\boldsymbol{\mathcal{J}} && + && u^{-2}\mathcal{G}_K && \text{(8.5d)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Src}_1(q) &= u^{-(K+2)}\boldsymbol{\mathcal{G}}_K && \text{(8.6a)}\\
\mathbf{Src}_2(q) &= u^{-(K+2)}\boldsymbol{\mathcal{G}}_K && \text{(8.6b)}\\
\mathbf{Src}_3(q) &= u^{-(K+3)}\boldsymbol{\mathcal{G}}_K && \text{(8.6c)}
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{\bullet}{\mathbf{B}}{}^\mu(q) &= u^{-2}\mathcal{J} && \text{(8.7a)}\\
\overset{\bullet}{Q}_{1n}(q) &= u^{-1}\mathcal{J} && \text{(8.7b)}\\
\overset{\bullet}{Q}_{2n}(q) &= \mathcal{J} && \text{(8.7c)}\\
\overset{\bullet}{Q}_{3n}(q) &= u^{-2}\mathcal{J} && \text{(8.7d)}
\end{aligned}
$$

Remark 8.1. This proposition is a detailed examination of large $|u|$ behavior of the constituents of the symmetric hyperbolic system (8.3a). To convey its significance, it is helpful to suppress all but the $\frac{\partial}{\partial q^0}$ derivatives in (8.3a) and analyze the caricature scalar ordinary differential equation

$$
\mathbf{b}\big(u,f\big)\,\tfrac{\mathrm{d}}{\mathrm{d}u}f \;=\; \mathbf{q}\big(u,f\big)\,f + \mathbf{s}(u) \tag{8.8}
$$

In this remark, u plays the role of q^0. Suppose $f(u)$ is a solution to this equation on $(-\infty,T)$, $T<0$, with asymptotic data $\lim_{u\to-\infty} f(u) = 0$ and $\mathbf{b}(u,f(u)) > 0$. How can we estimate $f(u)$? For all $u_1 < T$,

$$
f(u_1) = \int_{-\infty}^{u_1}\mathrm{d}u\;\exp\left(\int_{u}^{u_1}\mathrm{d}s\,\frac{\mathbf{q}(s,f(s))}{\mathbf{b}(s,f(s))}\right)\frac{\mathbf{s}(u)}{\mathbf{b}(u,f(u))} \tag{8.9}
$$

If there were constants $A>0$ and $b>c>0$ with

$$
\mathbf{b}(u,f(u)) \ge b \qquad \mathbf{q}(u,f(u)) \le c|u|^{-1} \qquad |\mathbf{s}(u)| \le A|u|^{-2} \tag{8.10}
$$

for all $u \in (-\infty, T)$, then (8.9) would imply

$$\sup_{u\in(-\infty,T)} |f(u)| \leq \frac{A}{b-c}\,\frac{1}{|T|} \tag{8.11}$$

The apparent difficulty is that the functions in (8.10) depend on the solution $f(u)$. However, if it can be shown that a *strictly weaker* bound than (8.11), say (8.11) with A replaced by $2A$, implies (8.10), then an open-closed argument justifies (8.11). More precisely, one would first cutoff $-\infty$ by a finite value, argue by continuity, and then remove the cutoff.

To apply this reasoning, assume, in analogy with (8.3a), that $\mathbf{b}$, $\mathbf{q}$ are affine linear in f:

$$\begin{aligned} \mathbf{b}(u,f) \;&=\; \mathbf{b}(u,0) + \overset{\bullet}{\mathbf{b}}(u)\, f & \qquad \overset{\bullet}{\mathbf{b}}(u) &= (\tfrac{\partial}{\partial f}\mathbf{b})(u,0) \\ \mathbf{q}(u,f) \;&=\; \mathbf{q}(u,0) + \overset{\bullet}{\mathbf{q}}(u)\, f & \qquad \overset{\bullet}{\mathbf{q}}(u) &= (\tfrac{\partial}{\partial f}\mathbf{q})(u,0) \end{aligned}$$

Also, in analogy with (8.5a), (8.5b), (8.7a), (8.7b), assume that there is a constant $\epsilon > 0$, so that

$$\bigl|\mathbf{b}(u,0) - \not{\mathbf{b}}\bigr| \leq \epsilon |u|^{-1} \qquad \bigl|\mathbf{q}(u,0)\bigr| \leq \epsilon |u|^{-1} \qquad \bigl|\overset{\bullet}{\mathbf{b}}(u)\bigr| \leq |u|^{-1} \qquad \bigl|\overset{\bullet}{\mathbf{q}}(u)\bigr| \leq |u|^{-1}$$

For convenience, suppose $\not{\mathbf{b}} = 1$. The last inequality in (8.10) is an analog of (8.6a). If

$$\epsilon,\ A,\ |T|^{-1} \quad \text{are sufficiently small,} \tag{8.12}$$

then (8.11), with A replaced by $2A$, implies (8.10), with $b = \frac{1}{2}$ and $c = \frac{1}{4}$. It follows from an open-closed argument that (8.11) is a genuine estimate for $f(u)$.

To interpret (8.12) in the light of our analogy, observe that the generic symbols $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ in the second column in (8.5a), (8.5b) can be made small by making $\Psi(0)$ (equivalently, **DATA**) and the angular scaling parameter a small.

We conclude the present discussion with the following remarks:

- The analog of the step from (8.10) to (8.11) for the system (8.3a) is provided by the energy estimate.
- Neglecting $\not{Q}$ for the moment, (8.5d), (8.6c), (8.7d) are similar to (8.5b), (8.6a), (8.7b), since $|t|^{-1} + u^{-1}$ is $\mathcal{O}(u^{-2})$ as $u \to -\infty$ uniformly for $\underline{u}$ in a compact set. The interpretation of (8.5c), (8.6b), (8.7c) is different, because (8.8) is not the appropriate toy model problem for the equation satisfied by Ξ_2. In fact, Ξ_2 satisfies an ordinary differential equation along the *short* integral curves of L, so that less u decay is required.
- The inequality for $\mathbf{q}$ in (8.10) remains true if we add any non-positive constant to $\mathbf{q}$. Analogously, the matrix $\not{Q}(t)$, see **(S7)**, appearing in (8.5c), (8.5d), has only non-positive eigenvalues. This is implicitly exploited in the proof of the energy estimate.

Remark 8.2. In (8.6a) all but the last component of $\mathbf{Src}_1$ are actually $u^{-(K+3)}\boldsymbol{\mathcal{G}}_K$. It is to accommodate the last component that we truncated the formal power series $[\,\Psi\,]$ at $K+1$ rather than K, see **(S2)**.

Proof (of Proposition 8.1). The proof is by direct verification, using **(S4)**, **(S6)**, **(S7)**, Proposition 5.2, Remark 5.4 and Definition 5.3. See the Supplement to Proposition

8.1 (Appendix G). To give the flavor, we schematize the calculations for a few representative cases. Let C be the complex conjugation operator. Now

matrix	component	
$\mathbf{B}^1(q,0) - \not{\mathbf{B}}^1$	$(4,5)$	$-\frac{1}{u}\mathbf{e} + \frac{1}{u^2}\mathcal{G}_K$
$Q_{22}(q,0) - \not{Q}_{22}(q)$	$(6,5)$	$-\omega_1(0)\, C - \overline{\omega_1(0)} + \frac{1}{u}\mathcal{G}_K$
$Q_{33}(q,0) - \not{Q}_{33}(q)$	$(1,1)$	$(\frac{2}{u} + \frac{1}{u^2}\mathcal{G}_K) - (-\frac{2}{\lvert t\rvert})$

in agreement with (8.5a), (8.5c) and (8.5d).

To verify (8.6a), note that $\mathbf{Src}_1 = \mathcal{G}_K$ has no constant term as a polynomial in $\Psi(k)$ and its first derivatives. This follows directly from the definition of **Src** and the properties of f given in **(S4)**. If S is replaced by S_K, see definition (8.1), then $\mathbf{Src}_1 = \boldsymbol{\mathcal{G}}_K$. There is an overall $u^{-(K+2)}$, by construction of the formal power series solution $[\,\Psi\,]$. This implies (8.6a). $\square$

To put (5.7b) in the form required by Proposition 7.7, we use

$\widehat{\text{(S1)}}$ Let $\Xi^\sharp = (\Xi^\sharp{}_1,\ \Xi^\sharp{}_2,\ \Xi^\sharp{}_3)$ be the field given by

$$
\begin{aligned}
&(\Xi^\sharp{}_1,\ \Xi^\sharp{}_2,\ \Xi^\sharp{}_3) \\
&= (y_1, y_2, y_3) \oplus (s_1, s_2, s_4, s_5, p_1, p_2, p_3, p_4, p_7, p_8) \oplus (s_3, p_5, p_6, p_9)
\end{aligned}
$$

where $\Psi^\sharp = (s,p,y)$ is the constraint field. There is a permutation matrix $\widehat{\pi}$ so that

$$
(s,p,y) \ = \ \widehat{\pi}\,(\Xi^\sharp{}_1,\ \Xi^\sharp{}_2,\ \Xi^\sharp{}_3).
$$

The field $\Xi^\sharp = (\Xi^\sharp{}_1, \Xi^\sharp{}_2, \Xi^\sharp{}_3)$ takes values in $\widehat{\pi}^{-1}\widehat{\mathcal{R}} \ \subset \ \mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$.

$\widehat{\text{(S2)}}$ System (5.7b) is abbreviated as $\widehat{\mathbf{A}}(q,\Psi)\Psi^\sharp \ = \ \widehat{\mathbf{f}}(q,\Psi,\partial_q\Psi)\Psi^\sharp$ (see, Convention 8.1). Some of its properties are discussed in Remark 5.3. System (5.7b) is equivalent to the linear, homogeneous symmetric hyperbolic system

$$
\begin{aligned}
\widehat{\mathbf{B}}(q,\Psi)\,\Xi^\sharp &= \widehat{Q}(q,\Psi,\partial_q\Psi)\,\Xi^\sharp \quad , \quad \widehat{\mathbf{B}} = \widehat{\mathbf{B}}^\mu \tfrac{\partial}{\partial q^\mu} \\
\widehat{\mathbf{B}}^\mu(q,\Psi) &= \widehat{\pi}^{-1}\,\widehat{\mathbf{A}}^\mu(q,\Psi)\,\widehat{\pi} \\
\widehat{Q}(q,\Psi,\partial_q\Psi) &= \widehat{\pi}^{-1}\,\widehat{\mathbf{f}}(q,\Psi,\partial_q\Psi)\,\widehat{\pi}
\end{aligned}
$$

The transformation $\widehat{Q}$ acting on $\widehat{\pi}^{-1}\widehat{\mathcal{R}}$ is linear over $\mathbb{R}$. Moreover, $\widehat{\mathbf{B}}^\mu$ depends affine linearly over $\mathbb{R}$ on Ψ, and $\widehat{Q}$ depends affine linearly over $\mathbb{R}$ on $\Psi \oplus \partial_q\Psi$.

$\widehat{\text{(S3)}}$ The three by three $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block-decomposition of $\widehat{\mathbf{B}}$ is

$$
\widehat{\mathbf{B}} = \operatorname{diag}\big(\widehat{B}_1, \widehat{B}_2, \widehat{B}_3\big), \qquad \widehat{B}_2 = \mathbb{1}_{10}\, L, \qquad \widehat{B}_3 = \mathbb{1}_4\, N,
$$

and $\widehat{B}_1$ is the 3×3 Hermitian matrix operator on the left hand side of (5.12c).

$\widehat{\text{(S4)}}$ $\widehat{\not{Q}}_1$, $\widehat{\not{Q}}_2$, $\widehat{\not{Q}}_3$ are constant 10×3, 4×10, 4×4 matrices. Their nonzero entries are (C is the complex conjugation operator):

$$
\begin{aligned}
&(\widehat{\not{Q}}_1)_{5,1} = 1 \qquad && (\widehat{\not{Q}}_2)_{1,9} = (\widehat{\not{Q}}_2)_{4,10} = -1 \\
&(\widehat{\not{Q}}_3)_{1,1} = -1 \qquad && (\widehat{\not{Q}}_2)_{1,10} = (\widehat{\not{Q}}_2)_{2,8} = (\widehat{\not{Q}}_2)_{3,7} = (\widehat{\not{Q}}_2)_{4,9} = -C
\end{aligned}
$$

Note that $\widehat{\not{Q}}_3 \leq 0$. Let $\widehat{\not{Q}}(t) = (\widehat{\not{Q}}_{mn})_{m,n=1,2,3}$ be the $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block matrix

$$\widehat{\not{Q}}(t) = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{\not{Q}}_1 & 0 & 0 \\ 0 & |t|^{-1}\widehat{\not{Q}}_2 & |t|^{-1}\widehat{\not{Q}}_3 \end{pmatrix}.$$

$\widehat{\text{(S5)}}$ $\widehat{\not{\mathbf{B}}}^\mu$ are the constant, diagonal, $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block matrices

$$\widehat{\not{\mathbf{B}}} = \mathbb{1}_3\, U \oplus \mathbb{1}_{10}\, V \oplus \mathbb{1}_4\, U \quad , \quad \widehat{\not{\mathbf{B}}} = \widehat{\not{\mathbf{B}}}^\mu \tfrac{\partial}{\partial q^\mu}$$

with U, V as in **(S8)**. Note that $\widehat{\not{\mathbf{B}}}^0 = \mathbb{1}_{17}$, $\widehat{\not{\mathbf{B}}}^1 = \widehat{\not{\mathbf{B}}}^2 = 0$, $\widehat{\not{\mathbf{B}}}^3 \geq 0$.

Definition 8.2. *Each entry to the left of the vertical bar is a generic symbol for a polynomial (with complex coefficients) in the (components of the) quantities to the right and their complex conjugates.*

$\mathcal{G}^\sharp \big|\; u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),\ \Psi - \Psi(0),$ *and first derivatives*

$\mathcal{G}^\sharp_0 \big|\; u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),\ \Psi - \Psi(0)$

$\mathcal{G}^\sharp_1 \big|$ *like* $\mathcal{G}^\sharp$*, but it has no constant term as a polynomial in* $\Psi - \Psi(0)$, $\partial_q\big(\Psi - \Psi(0)\big)$

$\boldsymbol{\mathcal{G}}^\sharp \big|\; u^{-1},\ \mathfrak{A},\ \underline{u},\ S_0,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),$ *and first derivatives*

where S_0 *is defined by* $S = -\mathfrak{A}^2\underline{u} + u^{-1}S_0$*, see* (4.4) *and* (5.4)*.*

Proposition 8.2. *Suppose* $\widehat{\text{(S1)}}$ *to* $\widehat{\text{(S5)}}$*. Then*

$$\begin{aligned}
\widehat{\mathbf{B}}^\mu(q,\Psi) &= \widehat{\not{\mathbf{B}}}^\mu && && + u^{-2}\mathcal{G}^\sharp_0 && \text{(8.13a)}\\
\widehat{Q}_{1n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{1n}(q) && && + u^{-2}\mathcal{G}^\sharp && \text{(8.13b)}\\
\widehat{Q}_{2n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{2n}(q) && + \boldsymbol{\mathcal{H}} && + u^{-1}\mathcal{G}^\sharp && \text{(8.13c)}\\
\widehat{Q}_{3n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{3n}(q) && + (|t|^{-1} + u^{-1})\boldsymbol{\mathcal{J}} && + u^{-2}\mathcal{G}^\sharp && \text{(8.13d)}
\end{aligned}$$

Moreover,

$$\Xi^\sharp{}_1,\ \Xi^\sharp{}_3,\ s_1,\ s_2,\ p_1,\ p_2,\ p_3 = u^{-1}\boldsymbol{\mathcal{G}}^\sharp + \quad \mathcal{G}^\sharp_1 \tag{8.14a}$$

$$s_4,\ s_5,\ p_4,\ p_7,\ p_8 = u^{-1}\boldsymbol{\mathcal{G}}^\sharp + u\,\mathcal{G}^\sharp_1 \tag{8.14b}$$

Finally, it is a consequence of (5.7b) *that*

$$L\begin{pmatrix} s_4 \\ s_5 \\ p_4 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} \mathbf{e}(\overline{p}_4 - p_5) + \ \ \mathbf{e}(\overline{p}_6 - p_3) \\ i\,\mathbf{e}(\overline{p}_4 - p_5) - i\,\mathbf{e}(\overline{p}_6 - p_3) \\ 0 \\ \boldsymbol{\lambda}(p_6 - \overline{p}_3) - \overline{\boldsymbol{\lambda}}(p_4 - \overline{p}_5) \\ -\overline{\boldsymbol{\lambda}}(\overline{p}_6 - p_3) + \boldsymbol{\lambda}(\overline{p}_4 - p_5) \end{pmatrix} + u^{-1}\,\mathcal{G}^\sharp\,\Xi^\sharp \tag{8.15}$$

Proof. The first part, (8.13), and the last part, (8.15), follow directly from Proposition 5.4, Remark 5.4 and Definition 5.3. It is entirely similar to the proof of Proposition 8.1.

To prove (8.14), write $\Psi = \Psi(0) + (\Psi - \Psi(0))$, and consider each object on the left hand side of (8.14) as a polynomial in $\Psi - \Psi(0)$ and $\partial_q(\Psi - \Psi(0))$, with coefficients possibly depending on $\Psi(0)$ and $\partial_q\Psi(0)$ (see, Proposition 5.3). The idea is that the constant term of this polynomial is of the generic form $u^{-1}\boldsymbol{\mathcal{G}}^\sharp$. Everything else is of the form $\mathcal{G}_1^\sharp$ or $u\,\mathcal{G}_1^\sharp$, respectively. The fact that the constant term is of the generic form $u^{-1}\boldsymbol{\mathcal{G}}^\sharp$ is an essential part of the construction. It is the fact that $\Psi(0)$ is built so that the first term in the formal power series of the constraint field, $\Psi^\sharp(0)$, vanishes (this follows from the vanishing of the formal constraint field and Remark 6.1). $\square$

Everything we have done in this section so far was to prepare for

Proposition 8.3 (Main Technical Proposition). *Fix $K \geq 0$ as in* **(S2)**. *Suppose* **(S1)** *through* **(S11)** *and* $\widehat{\textbf{(S1)}}$ *through* $\widehat{\textbf{(S5)}}$ *all hold. Let $R \geq 4$ be an integer. Set*

$$Y = \Big(\, R,\; K,\; \|\textbf{DATA}\|_{C^{R+2K+6}(\mathcal{Q})} \Big) \tag{8.16}$$

Let $\mathfrak{d}$ be as in (7.40). *Fix $\mathbf{c}_2' \in (0,1)$ and $T \in (-\infty, -\mathfrak{d}^{-1})$. There are constants $\mathbf{c}_6(R) \in (0,1)$ and $\mathbf{c}_7(Y) \in (0,1)$, non-increasing in all their arguments, such that Parts 1, 2 and 3 below hold whenever*

$$|a| \leq \mathbf{c}_6(R)\,\mathbf{c}_2' \quad , \quad \|\textbf{DATA}\|_{C^{R+4}(\mathcal{Q})} \leq \mathbf{c}_6(R)\,\mathbf{c}_2' \quad , \quad |T|^{-1} \leq \mathbf{c}_7(Y)\,\mathbf{c}_2' \tag{8.17}$$

Part 1. *The system $\mathbf{M}(q,\Xi)\,\Xi = h(q,\Xi)$ in* **(S10)** *satisfies* **(EB0)** *through* **(EB4)** *and the assumptions of Part 2 of Proposition 7.3, with:*

(EB0) - (EB4)	T	P	$\mathbf{M}^\mu(q,\Theta)$	$h(q,\Theta)$	$\not{\mathbf{M}}^\mu$	$\not{H}(t)$	$\mathcal{Q}$	$\mathcal{K}$
(S1) - (S10)	T	31	$\mathbf{M}^\mu(q,\Xi)$	$h(q,\Xi)$	$\not{\mathbf{B}}^\mu$	$\not{\mathcal{Q}}(t)$	$\mathcal{Q}$	$\mathcal{K}$

The table indicates that the symbols in the first row, appearing in the general **(EB0)** *through* **(EB4)**, *are given by the specific objects in the second row, appearing in* **(S1)** *through* **(S10)**.

Part 2. *If $t_0 < T$ and $\Xi : [t_0, t_0+\epsilon) \times \mathbb{R}^3 \to \pi^{-1}\mathcal{R}$ ($\epsilon > 0$) is a C^∞ solution to $\mathbf{M}(q,\Xi)\,\Xi = h(q,\Xi)$ which vanishes identically at t_0, then*

$$|t_0|^{K+1} \sup_{\xi_0 \in \mathbb{R}^2} \sqrt{E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\Xi\}(t_0)} \;\;\leq\;\; (\mathbf{c}_7(Y))^{-1} \tag{8.18}$$

Here, the energy $E^R_{\mathcal{O}(\xi,b,t)}$ is defined as in the Refined Energy Estimate, Proposition 7.7.

Part 3. *We distinguish three alternative systems, denoted by* **(Sys1)**, **(Sys2)** *and* **(Sys3)**, *that are given in the columns of the table below. This part of the Proposition applies to each of these systems individually. To evaluate the entries in the table, we require the following information. First,*

(Sys1):	$b = 2$	$\xi_0 \in \mathbb{R}^2$
(Sys2):	$b = 1$	$\xi_0 \in D_{2\|\frac{a}{\mathfrak{A}}\|}(0)$
(Sys3):	$b = 1$	$\xi_0 \in D_{2\|\frac{a}{\mathfrak{A}}\|}(0)$

Second, fix $t_0 < T$ and define the open set

$$\mathcal{V} \ = \ \bigcup_{t \in (t_0, T)} \{t\} \times \mathcal{O}(\xi_0, b, t) \ \ \subset \ \ \mathbb{R}^4.$$

For **(Sys1)** *and* **(Sys3)** *there is a single field $\varXi$ defined on $\mathcal{V}$ taking values in $\pi^{-1}\mathcal{R}$. For system* **(Sys2)***, there are two fields, $\varXi^{(1)}$ and $\varXi^{(2)}$, of this kind. The various fields satisfy the conditions:*

(i) They are C^p and their derivatives of order $\leq p$ extend continuously to $\overline{\mathcal{V}}$. Here $p = \infty$ for **(Sys1)**, **(Sys3)** *and $p = 1$ for* **(Sys2)**.
(ii) They are solutions to

$$\begin{cases} \mathbf{M}(q, \varXi)\varXi = h(q, \varXi) & \textit{for } \textbf{(Sys1)} \\ \mathbf{B}(q, \varXi^{(i)})\varXi^{(i)} = Q(q, \varXi^{(i)}) + \mathbf{Src}(q) & \textit{for } \textbf{(Sys2)} \\ \mathbf{B}(q, \varXi)\varXi = Q(q, \varXi) + \mathbf{Src}(q) & \textit{for } \textbf{(Sys3)} \end{cases}$$

See **(S4)** *and* **(S10)**.
(iii) They vanish when $q^3 < \frac{1}{2}$.
(iv) For all $t \in (t_0, T)$, they satisfy

$$\begin{cases} E^R_{\mathcal{O}(\xi_0,2,t)}\{\varXi\}(t) \ \leq \ (\mathbf{c}_6(R)\mathbf{c}_2')^2 & \textit{for } \textbf{(Sys1)} \\ \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi^{(i)}\}(t) \ \leq \ \mathbf{c}_6(R)\mathbf{c}_2' & \textit{for } \textbf{(Sys2)} \\ \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi\}(t) \ \leq \ \mathbf{c}_6(R)\mathbf{c}_2' & \textit{for } \textbf{(Sys3)} \end{cases}$$

To state the conclusions, recall the notation $\varUpsilon = \varXi^{(2)} - \varXi^{(1)}$ and the usage $\varPsi^\sharp$ for the constraint field associated to $\varPsi = \varPsi_K + \pi\,\varXi$, see **(S2)***. Finally, $\varPsi^\sharp = \widehat{\pi}\,\varXi^\sharp$, as in* $\widehat{\textbf{(S1)}}$.

Conclusion 1: $\varXi(\mathcal{V}),\ \varXi^{(1)}(\mathcal{V}),\ \varXi^{(2)}(\mathcal{V}) \ \subset \ \overline{B_{1/2}(0)} \ \subset \ \pi^{-1}\mathcal{R} \ \cong \ \mathbb{R}^{31}$.

Conclusion 2: *The assumptions* **(RE0)** *through* **(RE12)** *hold, with* **(RE11a)** *for* **(Sys1)** *and* **(RE11b)** *for* **(Sys2)** *and* **(Sys3)***, provided that the symbols in the first column of the table below (appearing in the general* **(RE0)** *through* **(RE12)***) are given by the specific objects in the other three columns.*

(RE0) - (RE12)	**(Sys1)**	**(Sys2)**	**(Sys3)**
$\mathcal{I} = (t_0, t^*)$	(t_0, T)	(t_0, T)	(t_0, T)
b	2	1	1
(P_1, P_2, P_3)	$(10, 15, 6)$	$(10, 15, 6)$	$(6, 18, 8)$
$\mathbf{M}^\mu(q)$	$\mathbf{M}^\mu(q, \varXi(q))$	$\mathbf{B}^\mu(q, \varXi^{(1)}(q))$	$\widehat{\mathbf{B}}^\mu(q, \varPsi(q))$
$H(q)$	$H(q, \varXi(q))$	$G\big(q, \varXi^{(1)}, \varXi^{(2)}, \partial_q \varXi^{(2)}\big)$	$\widehat{Q}(q, \varPsi, \partial_q \varPsi)$
$\mathbf{Src}(q)$	$\psi\,\mathbf{Src}(q)$	0	0
$\varTheta(q)$	$\varXi(q)$	$\varUpsilon(q)$	$\varXi^\sharp(q)$
$\not{\mathbf{M}}^\mu$	$\not{\mathbf{B}}^\mu$	$\not{\mathbf{B}}^\mu$	$\widehat{\not{\mathbf{B}}}^\mu$
$\not{H}(t)$	$\not{Q}(t)$	$\not{Q}(t)$	$\widehat{\not{Q}}(t)$
R	R	0	0
$\mathbf{c}_1$	$(\mathbf{c}_7(Y))^{-1}$	0	0
$\mathbf{c}_2$	$\mathbf{c}_2'$	$\mathbf{c}_2'$	$\mathbf{c}_2'$
J	$K + 1$	> 0	> 0
ξ_0	ξ_0	ξ_0	ξ_0

Conclusion 3: *For* **(Sys3)***, if in addition* $t_0+1<T$*, then*

$$\sup_{t\in(t_0+1,T)} |t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi^\sharp\}(t) \;\lesssim_Y\; \Big(1+\sup_{t\in(t_0,T)}|t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi\}(t)\Big) \quad (8.19)$$

Proof. We begin with a warning.

First Warning. In the course of this proof, we produce a finite chain of smallness assumptions on $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$. It is *essential*, for the purpose of showing that the far field expansion is truly an asymptotic expansion to a classical solution of (5.7a), that these smallness assumptions depend only on R and Y, respectively. To give a representative example, suppose **quantity** $\lesssim_R\; \mathbf{c}_6(R)$. Then there is a legitimate smallness assumption on $\mathbf{c}_6(R)$ making, say, **quantity** ≤ 1. By contrast, there is no legitimate smallness assumption associated to **quantity** $\lesssim_Y\; \mathbf{c}_6(R)$. We can take a more relaxed attitude to the system (5.7b), because it is only necessary to demonstrate uniqueness.

Convention 8.5. In this proof, the constants of proportionality in $\lesssim_R$ and $\lesssim_Y$ are always non-decreasing in R and the components of Y, respectively.

Overall Preliminaries. For all $n\geq 0$ and $0\leq k\leq K+1$ and $\beta\in\mathbb{N}_0^4$ with $|\beta|\leq 1+R$, the following estimates hold on $(-\infty,T)\times\mathcal{Q}$:

$$\begin{array}{lll}
|\partial^\beta u^{-n}| \lesssim_{(R,n)} |t|^{-n} & |\partial^\beta \mathfrak{A}| = |\mathfrak{A}|\,\delta_{\beta 0}\leq |a| & |\partial^\beta \underline{u}|\leq 2\\
|\partial^\beta \Psi(0)| \lesssim_R \mathbf{c}_6(R)\,\mathbf{c}_2' & |\partial^\beta \mathbf{e}|\leq \tfrac{17}{2}\,|a| & |\partial^\beta \boldsymbol{\lambda}|\leq \tfrac{17}{2}\,|a|\\
|\partial^\beta \Psi(k)| \lesssim_Y 1 & |\partial^\beta S| \lesssim_R 1 & |\partial^\beta S_K|\lesssim_Y 1
\end{array}$$

Only the estimates on $\Psi(0)$ and $\Psi(k)$ require discussion. They follow from Proposition 6.3, and (8.16) and (8.17).

It follows from the estimates just above, the product rule and (8.17), that for all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\leq R$,

$$\begin{array}{lrll}
n=0,1 & |t|^n\,|\partial^\alpha(u^{-(n+1)}\mathcal{G}_K)| & \lesssim_Y\; |T|^{-1}\lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}_2' & (8.20a)\\
n=1,2 & |t|^{K+n}|\partial^\alpha(u^{-(K+n)}\boldsymbol{\mathcal{G}}_K)| & \lesssim_Y\; 1 & (8.20b)\\
n=0,1 & |t|^n\,|\partial^\alpha(u^{-n}\mathcal{H})| & \lesssim_R\; \mathbf{c}_6(R)\,\mathbf{c}_2' & (8.20c)\\
n=0,1 & |t|^n\,|\partial^\alpha(u^{-n}\boldsymbol{\mathcal{H}})| & \lesssim_R\; |a|\lesssim_R \mathbf{c}_6(R)\,\mathbf{c}_2' & (8.20d)\\
n=0,1,2 & |t|^n\,|\partial^\alpha(u^{-n}\mathcal{J})| & \lesssim_R 1 & (8.20e)\\
 & |t|\,|\partial^\alpha(|t|^{-1}+u^{-1})\boldsymbol{\mathcal{J}}| & \lesssim_R\; |T|^{-1}\lesssim_R \mathbf{c}_7(Y)\,\mathbf{c}_2' & (8.20f)
\end{array}$$

at every point of $(-\infty,T)\times\mathcal{Q}$. In this instance, the constants also depend on the particular polynomial represented by the generic symbols. Observe that in the second inequality, one does not use the property that $\boldsymbol{\mathcal{G}}_K$ has no constant term as a polynomial in $\Psi(k)$ and its derivatives.

Second Warning. It is crucially important that whenever $\lesssim_R$ appears in an estimate (for example, (8.20c), (8.20d), (8.20e), (8.20f)) that the generic symbol on the left hand side has no subindex K, see Definition 8.1. On the other hand, whenever $\lesssim_Y$ appears (for example, (8.20a), (8.20b)), the generic symbol on the left hand side is allowed to carry a subindex K.

Preliminaries for Part 3. For Part 3, it is necessary to supplement the Overall Preliminaries. Let $(\mathcal{V}_1, \mathcal{V}_2)$ be the open cover of $\mathcal{V}$ given by

$$\mathcal{V}_1 = \mathcal{V} \cap \big((t_0, T) \times \mathcal{Q}\big), \qquad \mathcal{V}_2 = \mathcal{V} \cap \big((t_0, T) \times (\mathbb{R}^3 \setminus \mathcal{K})\big).$$

The sets $\mathcal{Q}$, $\mathcal{K}$ are defined in **(S9)**.

- For **(Sys1)**, observe that the Overall Preliminaries apply to $\mathcal{V}_1$. On $\mathcal{V}_2$, we have $\psi = 0$, and the equations simplify, see **(S10)**. The estimate $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$, see **(S9)**, will be used on the transition region for ψ.
- For **(Sys2)** we have $\mathcal{V} = \mathcal{V}_1$. In this case, the Overall Preliminaries will suffice.
- For **(Sys3)** we also have $\mathcal{V} = \mathcal{V}_1$. However, in addition to the Overall Preliminaries, we require the estimates

$$\begin{aligned} |\varPsi - \varPsi(0)| \ \ , \ \ \big|\partial_q\big(\varPsi - \varPsi(0)\big)\big| \ &\lesssim_Y \ 1 \\ |t|\,|u^{-2}\mathcal{G}^\sharp_0| \ \ , \ \ |t|\,|\partial_q(u^{-2}\mathcal{G}^\sharp_0)| \ &\lesssim_Y \ |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}'_2 \\ n = 0, 1: \qquad |t|^n\,|u^{-(n+1)}\mathcal{G}^\sharp| \ &\lesssim_Y \ |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}'_2, \\ |\boldsymbol{\mathcal{H}}| \ &\lesssim \ |a| \lesssim \mathbf{c}_6(R)\,\mathbf{c}'_2 \\ |t|\,|(|t|^{-1} + u^{-1})\boldsymbol{\mathcal{J}}| \ &\lesssim_Y \ |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}'_2. \end{aligned} \tag{8.21}$$

  on $\mathcal{V}$. Estimate (8.21) follows from

$$\varPsi - \varPsi(0) = \sum_{k=1}^{K+1} (\tfrac{1}{u})^k\,\varPsi(k) + \pi\,\varXi \tag{8.22}$$

  and condition (iv) in the Proposition. The rest are consequences of (8.21) and the Overall Preliminaries estimates.

Proof of Part 1. One verifies **(EB0)** and **(EB4)** by direct inspection, apart from the inequality in **(EB1)**. However,

$$\tfrac{1}{2} \le \mathbf{M}^0(q, \varXi) \le 2, \qquad \mathbf{M}^3(q, \varXi) \ge 0, \tag{8.23}$$

for all $(q, \varXi) \in (-\infty, T) \times \mathbb{R}^3 \times B_2(0)$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are small enough. Here $B_2(0) \subset \mathbb{R}^P$ as in **(EB0)**, with $P = 31$. The first inequality in (8.23) completes the verification of **(EB1)**. The second inequality in (8.23) is used for the first supplemental hypothesis in Part 2 of Proposition 7.3.

To check (8.23), recall from **(S10)** that

$$\mathbf{M}^\mu(q, \varXi) = \psi\,\mathbf{B}^\mu(q, \varXi) + (1 - \psi)\,\not{\mathbf{B}}^\mu \tag{8.24}$$

is a convex combination of $\mathbf{B}^\mu$ and $\not{\mathbf{B}}^\mu$ on $(-\infty, T) \times \mathbb{R}^3 \times B_2(0)$. It suffices to verify (8.23) for $\mathbf{B}^\mu$ and $\not{\mathbf{B}}^\mu$ separately. For $\not{\mathbf{B}}^\mu$, see **(S8)**. For $\mathbf{B}^\mu$ it suffices to verify

$$\tfrac{1}{2} \le 1 + \tfrac{1}{u^2} f_3 \le 2, \qquad \tfrac{1}{2} \le 1 + \tfrac{1}{u^2}(1 + \tfrac{1}{u^2} f_3) \le 2 \tag{8.25}$$

for $\mathbf{q} \in \mathrm{supp}_{\mathbb{R}^3}\,\psi \subset \mathcal{Q}$, see **(S6)** and Remark 5.4. Here, f_3 is one of the components of $\varPsi = \varPsi_K + \pi\,\varXi = (f, \omega, z)$, see **(S3)**. To check (8.25), note that

$$|\varPsi| \ \le \ |\varPsi(0)| + \tfrac{1}{|u|} \sum_{k=1}^{K+1} \tfrac{1}{|u|^{k-1}}\,|\varPsi(k)| + |\varXi|.$$

The three terms are respectively $\lesssim_R \mathbf{c}_6(R)$ and $\lesssim_Y \frac{1}{|T|} \lesssim_Y \mathbf{c}_7(Y)$ (see, Overall Preliminaries) and ≤ 2. By the choice of $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ (see, the First Warning), we can make

$$|\Psi| \leq 3 \ \ , \text{ when } \quad (q, \Xi) \in (-\infty, T) \times \mathcal{Q} \times B_2(0) \tag{8.26}$$

Consequently, $|f_3| \leq 3$, and therefore, (8.25) holds because $\frac{1}{|u|} \leq \frac{1}{|T|} \leq \mathfrak{d}$, see (7.40).

To validate the second supplemental hypothesis in Part 2 of Proposition 7.3, it is necessary to show that $h(q,0) = \psi\,\mathbf{Src}(q) = 0$ for all $q \in (-\infty, T) \times \mathbb{R}^3$ with $q^3 < \frac{1}{2}$. To do this, observe that $\Psi_K = 0$ there, see **(S2)**.

Remark 8.3. Later on, in the proof of Part 3, we need the analogous inequalities

$$\tfrac{1}{2} \leq \widehat{\mathbf{B}}^0(q, \Psi) \leq 2, \qquad \widehat{\mathbf{B}}^3(q, \Psi) \geq 0$$

for all $(q, \Xi) \in (-\infty, T) \times \mathbb{R}^3 \times B_2(0)$, with the same smallness assumptions. These inequalities can again be reduced to (8.25).

Proof of Part 2. To prove (8.18), rewrite $\mathbf{M}(q, \Xi)\,\Xi = h(q, \Xi)$ as

$$\partial_t \Xi(q) = \big(\mathbf{M}^0(q, \Xi)\big)^{-1} \Big(- \sum_{i=1,2,3} \mathbf{M}^i(q, \Xi)\partial_i \Xi + H(q, \Xi)\,\Xi + \psi\,\mathbf{Src} \Big) \tag{8.27}$$

Here, $\mathbf{M}^0(\,\cdot\,, \Xi)$ is invertible on an open neighborhood of $\{t_0\} \times \mathbb{R}^3$ in the set $[t_0, t_0 + \epsilon) \times \mathbb{R}^3$. This is a consequence of (8.23), and the assumption $\Xi(t_0, \,\cdot\,) \equiv 0$.

By repeated differentiation of (8.27) with respect to t, we obtain an expression for $\partial_t^m \Xi(q)$, for any $m \geq 1$. Restrict the result to $\{t_0\} \times \mathbb{R}^3$ and simplify it using the assumption $\partial^\beta \Xi(t_0, \,\cdot\,) \equiv 0$ for all $\beta \in \mathbb{N}_0^4$ with $\beta_0 = 0$. In every surviving multiderivative $\partial^\beta \Xi(t_0, \,\cdot\,)$ we must have $1 \leq \beta_0 \leq m-1$, and each one is recursively expressed using $\partial_t^n \Xi(t_0, \,\cdot\,)$ with $1 \leq n \leq m-1$. This procedure generates an explicit expression for $\partial_t^m \Xi(t_0, \,\cdot\,)$ in terms of the quantities in (8.28) just below, and their derivatives.

By differentiation with respect to the remaining coordinates $\mathbf{q}$, we find (inductively) that for every $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq R$, the function $\partial^\alpha \Xi(t_0, \,\cdot\,)$ is a polynomial in $(\mathbf{M}^0((t_0, \,\cdot\,), 0))^{-1}$ as well as derivatives of order $\leq R-1$ of

$$\mathbf{M}^\mu\big((t_0, \,\cdot\,), 0\big) \ , \ H\big((t_0, \,\cdot\,), 0\big) \ , \ \overset{\bullet}{\mathbf{M}}{}^\mu(t_0, \,\cdot\,) \ , \ \overset{\bullet}{H}(t_0, \,\cdot\,) \ , \ \psi\,\mathbf{Src}(t_0, \,\cdot\,) \tag{8.28}$$

This polynomial has no constant term as a polynomial in $\psi\,\mathbf{Src}(t_0, \,\cdot\,)$ and its derivatives. This has two consequences. Consequence A, $\partial^\alpha \Xi(t_0, \,\cdot\,)$ vanishes on $\mathbb{R}^3 \setminus \mathcal{K}$, by the support of ψ. Consequence B,

$$|t_0|^{K+1} \left|\partial^\alpha \Xi(t_0, \mathbf{q})\right| \lesssim_Y 1$$

for all $\mathbf{q} \in \mathcal{Q}$. To verify this inequality, check (using $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$ in **(S9)**, the first inequality in (8.23), Proposition 8.1 and the Overall Preliminaries) that the matrix $(\mathbf{M}^0((t_0, \,\cdot\,), 0))^{-1}$ and the derivatives of order $\leq R-1$ of all the terms in (8.28) are bounded in absolute value on $\mathcal{Q}$ by $\lesssim_Y 1$. At this point, we have $\left|\partial^\alpha \Xi(t_0, \mathbf{q})\right| \lesssim_Y 1$. To get the stated decay, use (8.6) (even though one can get a better result). It is here that one exploits the fact that the expression for $\partial^\alpha \Xi(t_0, \,\cdot\,)$ has no constant term as a polynomial in $\psi\,\mathbf{Src}(t_0, \,\cdot\,)$ and its derivatives.

The proof of Part 2 is completed by combining Consequences A and B with

$$E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\varXi\}(t_0) \lesssim_R \big(\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,2,t_0)}\{\varXi\}(t_0)\big)^2$$

and making a suitable choice of $\mathbf{c}_7(Y)$ (see, the First Warning).

Proof of Part 3, Conclusion 1. Follows from condition (iv) in the Proposition, by suitable choice of $\mathbf{c}_6(R)$. For **(Sys1)**, we also use $R \geq 2$ and the Sobolev inequality (7.43).

Proof of Part 3, Conclusion 2. To start with, we check that **(RE10)** and **(RE11a)** or **(RE11b)** hold, when $\mathbf{c}_6(R)$, $\mathbf{c}_7(Y)$ are made sufficiently small. We will freely use the Overall Preliminaries, the Preliminaries for Part 3, Propositions 8.1, 8.2, and the inequality

$$E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) \lesssim_R \big(\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)\big)^2. \tag{8.29}$$

- **(Sys1)**: Let $t \in (t_0, T)$. For **(RE10)**, we have

  $$|t|^{2K+4}E^R_{\mathcal{O}(\xi_0,2,t)}\{\psi\,\mathbf{Src}_1\}(t) = |t|^{2K+4}E^R_{\mathcal{O}(\xi_0,2,t)}\{\psi\,u^{-(K+2)}\boldsymbol{\mathcal{G}}_K\}(t) \lesssim_Y 1$$

  Therefore, the left hand side is $\leq (\mathbf{c}_7(Y))^{-2}$, when $\mathbf{c}_7(Y) > 0$ is small enough (see, the First Warning). Similarly, for $\psi\,\mathbf{Src}_2$ and $\psi\,\mathbf{Src}_3$. In these two cases, one could get a better decay estimate, but we don't need it.
  For **(RE11a)**, we verify the first and second inequalities, the other two are similar. The second goes

  $$\begin{aligned}
  &|t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{H_{1n}(q,\varXi) - \not{Q}_{1n}\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(Q_{1n}(q,0) - \not{Q}_{1n}\big) + \psi\,\dot{Q}_{1n}(q)\varXi\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K\big) + \psi\,\tfrac{1}{u}\mathcal{J}\,\varXi\big\}(t)\\
  &\lesssim_R |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u}\mathcal{J}\,\varXi\big)\big\}(t) + |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\,\tfrac{1}{u^2}\mathcal{G}_K\big\}(t)
  \end{aligned}$$

  The first term is $\lesssim_R (\mathbf{c}_6(R)\mathbf{c}_2')^2$, the second is $\lesssim_Y (\mathbf{c}_7(Y)\mathbf{c}_2')^2$. Here, we use condition (iv) in the Proposition. By the First Warning, the result is $\leq (\mathbf{c}_2')^2$ if $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are small enough. The first goes

  $$\begin{aligned}
  &|t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\mathbf{M}^\mu(q,\varXi) - \not{\mathbf{B}}^\mu\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi(\mathbf{B}^\mu(q,0) - \not{\mathbf{B}}^\mu) + \psi\,\dot{\mathbf{B}}^\mu(q)\,\varXi\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\,(\tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K) + \psi\,\tfrac{1}{u^2}\mathcal{J}\,\varXi\big\}(t)
  \end{aligned}$$

  The estimate is completed just as above.
- **(Sys2)**: There is nothing to check for **(RE10)**. For **(RE11b)**,

  $$\begin{aligned}
  &|t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\mathbf{B}^\mu(q,\varXi^{(1)}) - \not{\mathbf{B}}^\mu\big\}(t)\\
  &= |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\big(\mathbf{B}^\mu(q,0) - \not{\mathbf{B}}^\mu\big) + \dot{\mathbf{B}}^\mu(q)\,\varXi^{(1)}\big\}(t)\\
  &= |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K + \tfrac{1}{u^2}\mathcal{J}\,\varXi^{(1)}\big\}(t)
  \end{aligned}$$

which is $\leq \mathbf{c}_2'$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. It is important here that $\mathcal{V} = \mathcal{V}_1$, see, the Preliminaries for Part 3. For the second, third and fourth parts of **(RE11b)**, observe that, by (8.4),

$$\begin{aligned}(G - \not{Q})\, \Pi \ &= \big(Q(q,0) - \not{Q}(t)\big)\Pi + \big(\overset{\bullet}{Q}(q)\, \Xi^{(1)}\big)\Pi \\ &\qquad + \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\Big(- \mathbf{B}^{\mu}(q, s\Pi)\, \tfrac{\partial \Xi^{(2)}}{\partial q^{\mu}} + Q(q, s\Pi)\, \Xi^{(2)}\Big).\end{aligned}$$

Therefore, for the second,

$$\begin{aligned}&|t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{G_{1n}\big(q, \Xi^{(1)}, \Xi^{(2)}, \partial_q \Xi^{(2)}\big) - \not{Q}_{1n}(t)\big\}(t) \\ &= |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K + \tfrac{1}{u}\mathcal{J}\,\Xi^{(1)} + \tfrac{1}{u^2}\mathcal{J}\,\partial_q \Xi^{(2)} + \tfrac{1}{u}\mathcal{J}\,\Xi^{(2)}\big\}(t)\end{aligned}$$

which is $\leq \mathbf{c}_2'$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. The third an fourth inequalities in **(RE11b)** are checked in the same way.

- **(Sys3)**: There is nothing to check for **(RE10)**. For **(RE11b)**,

$$\begin{aligned}|t|\, \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{\mathbf{B}}^{\mu}(q,\Psi) - \widehat{\not{\mathbf{B}}}^{\mu}\big\}(t) &= |t|\, \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u^2}\mathcal{G}^{\sharp}_0\big\}(t) \\ |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{1n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{1n}\big\}(t) &= |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u^2}\mathcal{G}^{\sharp}\big\}(t) \\ \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{2n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{2n}\big\}(t) &= \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\boldsymbol{\mathcal{H}} + \tfrac{1}{u}\,\mathcal{G}^{\sharp}\big\}(t) \\ |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{3n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{3n}\big\}(t)& \\ &= |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{(\tfrac{1}{|t|} + \tfrac{1}{u})\boldsymbol{\mathcal{J}} + \tfrac{1}{u^2}\mathcal{G}^{\sharp}\big\}(t)\end{aligned}$$

which are all $\leq \mathbf{c}_2'$, when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. It is important here that $\mathcal{V} = \mathcal{V}_1$, see, the Preliminaries for Part 3.

We are now finished checking **(RE10)** and **(RE11a)** or **(RE11b)**.

Next, we check **(RE1)**. In order, **(RE1)** follows from

- **(Sys1)**: inequalities (8.23), since $\Xi(q) \in \overline{B_{\frac{1}{2}}(0)} \subset \mathbb{R}^{31}$ for all $q \in \mathcal{V}$ by Conclusion 1.
- **(Sys2)**: the discussion of (8.25), Conclusion 1 and $\mathcal{V} = \mathcal{V}_1$.
- **(Sys3)**: Remark 8.3, Conclusion 1 and $\mathcal{V} = \mathcal{V}_1$.

To check **(RE12)**, we can assume $|\Psi| \leq 3$ on $\mathcal{V}_1$, by (8.26) and Conclusion 1. By Proposition 7.6 and **(S1)**, this implies **(RE12)** for **(Sys2)** and **(Sys3)**, because $\mathcal{V} = \mathcal{V}_1$. For **(Sys1)**, we use convexity, see (8.24). The $\psi\mathbf{B}$ term is again handled by Proposition 7.6, using the support properties of ψ. It therefore suffices to verify that

$$\theta_{\mu}\, \not{\mathbf{B}}^{\mu} \geq 0 \qquad \text{on} \qquad (\partial\mathcal{V}) \cap ((t_0, T) \times \mathbb{R}^2 \times (0,2))$$

with θ defined by (7.38). This is a consequence of $\theta(U) = k_1|\underline{u}_0 - \underline{u}|/|u|^2 \geq 0$ and $\theta(V) = k_1||u|^{-1} - |u_0|^{-1}| \geq 0$, see **(S8)** and (7.40).

(RE5) holds, by condition (i) in the Proposition, for **(Sys1)**, **(Sys2)** and **(Sys3)**. In particular, for **(Sys2)**, $G\big(q, \Xi^{(1)}, \Xi^{(2)}, \partial_q \Xi^{(2)}\big)$ is C^0.

(RE4) holds, by condition (ii) in the Proposition, for **(Sys1)**, **(Sys2)** and **(Sys3)**. For **(Sys2)**, see **(S11)**. For **(Sys3)**, recall that $\Psi^{\sharp}$ solves (5.7b) because Ψ solves (5.7a).

Note that condition (iii) in the Proposition and **(S2)** imply that $\Psi = \Psi_K + \pi\,\Xi = 0$ and $\Psi^{\sharp} = 0$ when $q^3 < \frac{1}{2}$. In particular, $\mathbf{Src} = 0$ there. These facts imply **(RE6)** for **(Sys1)**, **(Sys2)** and **(Sys3)**.

The remaining assumptions, **(RE0)**, **(RE2)**, **(RE3)**, **(RE7)**, **(RE8)**, **(RE9)** are verified by direct inspection.

Proof of Part 3, Conclusion 3. We have to prove

$$\sup_{t\in(t_0+1,T)} |t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi^\sharp\}(t) \;\lesssim_Y\; \Big(1+\sup_{t\in(t_0,T)} |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi\}(t)\Big) \;\stackrel{\text{def}}{=}\; \kappa$$

It follows from the Overall Preliminaries, the Preliminaries for Part 3, as well as (8.14) and (8.15) (see, Proposition 8.2) that on $\mathcal{V}=\mathcal{V}_1$,

(A)	$\lvert\boldsymbol{\mathcal{G}}^\sharp\rvert \lesssim_R 1$	by Definition 8.2
(B)	$\lvert\Psi-\Psi(0)\rvert,\ \ \big\lvert\partial_q(\Psi-\Psi(0))\big\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (8.22)
(C)	$\lvert\mathcal{G}_1^\sharp\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (B), Definition 8.2
(D)	$\lvert\varXi^\sharp{}_1\rvert,\lvert\varXi^\sharp{}_3\rvert,\lvert s_1\rvert,\lvert s_2\rvert,\lvert p_1\rvert,\lvert p_2\rvert,\lvert p_3\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (A), (C), (8.14a)
(E)	$\lvert\varXi^\sharp\rvert \lesssim_Y \kappa$	by (A), (C), (8.14a), (8.14b)
(F)	$\lvert u^{-1}\mathcal{G}^\sharp\varXi^\sharp\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (E), Definition 8.2

For each point $(t_1,\mathbf{q}_1)\in\mathcal{V}$ with $t_1>t_0+1$, consider the line segment

$$\mathbf{Seg} \;=\; \big((t_1,\mathbf{q}_1)-\mathbb{R}_+(1,0,0,1)\big)\cap\{q\in\mathbb{R}^4 \;\big|\; q^3>0\}.$$

We have $\mathbf{Seg}\subset\mathcal{V}$. To see this, view $\mathcal{V}$ as an open disk bundle over the $(t,\underline{u})$-rectangle $(t_0,T)\times(0,1)$. The projection of **Seg** to the $(t,\underline{u})$ plane is injective and contained in the base, because $t_1\in(t_0+1,T)$. At each point in the image of the projection of **Seg**, the corresponding point on **Seg** is contained in the fiber, because the radius function $r(\underline{u},u)$ (see (7.37)) is a decreasing function of $\underline{u}$ on the base for fixed $u=t-\underline{u}$, and because the endpoint $(t_1,\mathbf{q}_1)$ is contained in the fiber, by assumption.

By Conclusion 2, $\varXi^\sharp$ is a C^∞ solution to $\widehat{\mathbf{B}}\varXi^\sharp=\widehat{Q}\varXi^\sharp$ which vanishes when $q^3<\frac{1}{2}$. In particular (8.15) holds. Recall $L=e_3\big(\frac{\partial}{\partial q^0}+\frac{\partial}{\partial q^3}\big)$, where $\varPhi=(e,\gamma,w)$, and $\frac{1}{2}\le e_3\le 2$ (see **(RE1)**), and that **Seg** is an integral curve of L. The last three sentences, $\mathbf{Seg}\subset\mathcal{V}$ and (F) imply, by integrating the equation for p_4 in (8.15) along **Seg**, that

$$|p_4| \lesssim_Y \kappa\,|t|^{-1} \tag{8.30}$$

on **Seg** for each endpoint $(t_1,\mathbf{q}_1)\in\mathcal{V}$ with $t_1>t_0+1$.

Finally, integrating the remaining equations in (8.15) along **Seg**, and using (8.30) and (D), we obtain $|s_4|,\,|s_5|,\,|p_7|,\,|p_8|\;\lesssim_Y\;\kappa|t|^{-1}$ on all admissible segments **Seg**. Therefore, $|\varXi^\sharp|\lesssim_Y\kappa|t|^{-1}$ on $\mathcal{V}$ when $t\in(t_0+1,T)$. □

8.2. Construction of classical vacuum fields.

Theorem 8.1. *Let* $(\xi,\underline{u},u)$ *be the usual coordinates on the truncated strip*

$$\mathbf{Strip}(1,\lambda) \;=\; \mathbb{R}^2\times(0,1)\times\big(-\infty\,,\,-\lambda^{-1}\big)$$

of width 1*, for each* $\lambda>0$*. Suppose* $0<|\mathfrak{A}|\le|a|$*. Assume the functions*

$$\mathbf{DATA}^\sigma(\xi,\underline{u}):\ \mathbb{R}^2\times(0,\infty)\to\mathbb{C}$$

$\sigma \in \{-,+\}$, are smooth, vanish when $\underline{u} < \frac{1}{2}$, and are Pole-Flip compatible

$$\text{DATA}^{\sigma} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \text{DATA}^{-\sigma} \tag{8.31}$$

for all $(\xi, \underline{u}) \in (\mathbb{R}^2 \setminus \{0\}) \times (0,\infty)$, see Definition 6.2. Let $[\,\Psi^{\sigma}\,]$ be the formal power series solution corresponding to DATA^{σ}. Fix integers $R \geq 4$, $K \geq 0$ and an $\epsilon \in (0, \frac{1}{2})$. Set

$$B = (R,\epsilon) \quad , \quad C = \Big(R,\ \epsilon,\ K,\ \max_{\sigma\in\{-,+\}} \|\text{DATA}^{\sigma}\|_{C^{R+2K+6}\left(\mathcal{C}(a,\,\mathfrak{A},\,2)\right)}\Big)$$

where $\mathcal{C}(a,\mathfrak{A},b) = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0,b)$ for each $b > 0$.

Let

$$\mathbf{b} = \mathbf{b}(B) \qquad\qquad \mathbf{c} = \mathbf{c}(C)$$

be constants in $(0,1)$. If $\mathbf{b}$ and $\mathbf{c}$ are made sufficiently small depending only on B and C, respectively, then the Existence and Uniqueness statements below hold whenever

$$0 < |\mathfrak{A}| \leq |a| \leq \mathbf{b}, \qquad \max_{\sigma\in\{-,+\}} \|\text{DATA}^{\sigma}\|_{C^{R+4}(\mathcal{C}(a,\mathfrak{A},2))} \ \leq\ \mathbf{b} \tag{8.32}$$

Existence:
Part 1: *There exists a pair (Ψ^-, Ψ^+) of Pole-Flip compatible C^1-fields*

$$\Psi^{\sigma} : \mathbf{Strip}(1,\mathbf{c}) \ \to\ \mathcal{R},$$

which are both solutions to (5.7a)*, vanish when $\underline{u} < \frac{1}{2}$, extend with their first derivatives continuously to $\overline{\mathbf{Strip}(1,\mathbf{c})}$ and satisfy*

$$\lim_{u\to-\infty} |u|^{\epsilon} \sup_{\alpha\in\mathbb{N}_0^4:\ |\alpha|\leq 1} \Big\| \,\partial^{\alpha}\big(\Psi^{\sigma} - \Psi^{\sigma}(0)\big)(\,\cdot\,,u)\,\Big\|_{C^0\left(\mathcal{C}(a,\mathfrak{A},1)\right)} = 0 \tag{8.33}$$

Part 2: *The constraint fields $(\Psi^-)^{\sharp}$, $(\Psi^+)^{\sharp}$ associated to the fields in Part 1 vanish, and*

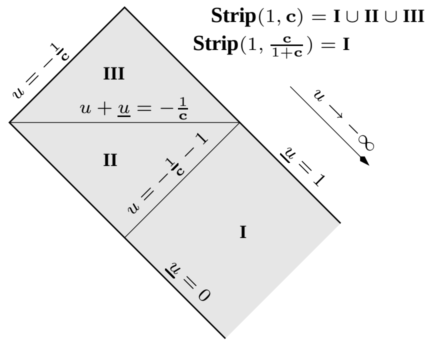
$$(\Phi^-, \Phi^+) \ =\ \big(\mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^-,\ \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^+\big)$$

are a pair of Pole-Flip compatible vacuum fields (see, Definition 2.2) with initial data $\Psi^{\sigma}(0)$.
Part 3: *The fields in Part 1 are actually of class C^{R-3}, and extend with their derivatives of order $\leq R-3$ continuously to $\overline{\mathbf{Strip}(1,\mathbf{c})}$. Moreover*

$$\sup_{u<-\mathbf{c}^{-1}} |u|^{K+1} \sup_{\substack{\alpha\in\mathbb{N}_0^4\\ |\alpha|\leq R-3}} \Big\|\partial^{\alpha}\Big(\Psi^{\sigma}(\,\cdot\,,u) - \sum_{k=0}^{K}\frac{\Psi^{\sigma}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(\mathcal{C}(a,\mathfrak{A},1)\right)} \ \leq\ \frac{1}{\mathbf{c}} \tag{8.34}$$

Uniqueness: *Assume (Ψ^-, Ψ^+) and $(\widetilde{\Psi}^-, \widetilde{\Psi}^+)$ have all the properties listed in Part 1. Then they coincide on $\mathbf{Strip}(1, \frac{\mathbf{c}}{1+\mathbf{c}}) \subset \mathbf{Strip}(1,\mathbf{c})$.*

Remark 8.4. Part 1 asserts the existence of a solution on $\mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$. However, Uniqueness in Theorem 8.1 refers only to $\mathbf{I}$. We actually prove uniqueness on $\mathbf{I} \cup \mathbf{II}$. By a standard finite speed of propagation argument, which we do not carry out, the domain of uniqueness can be extended to $\mathbf{III}$.



Remark 8.5. The **DATA**$^\sigma$ are given for $\underline{u} \in (0,\infty)$, just for convenience. By construction, the restriction of $\Psi^\sigma(0)$ to $\underline{u} \in (0,1)$ depends only on the restriction of **DATA**$^\sigma$ to $\underline{u} \in (0,1)$, see Definition 5.1. It now follows from Uniqueness in Theorem 8.1 that Ψ^-, Ψ^+ are determined on **Strip**$(1, \frac{\mathbf{c}}{1+\mathbf{c}})$ by the restriction of **DATA**$^\sigma$ to $\underline{u} \in (0,1)$.

Proof. Theorem 8.1 is formulated in the coordinate system $(\xi, \underline{u}, u)$. Almost the entire proof, however, is given in the coordinate system $q = (t, \xi, \underline{u})$, where $t = u + \underline{u}$.

We assume **(S1)** through **(S11)** and $\widehat{\textbf{(S1)}}$ through $\widehat{\textbf{(S5)}}$, where a, $\mathfrak{A}$, K in **(S1)** through **(S11)** are identified with their occurrences in the statement of Theorem 8.1 and where **DATA** in **(S2)** is identified with either one of **DATA**$^\sigma$, $\sigma = -, +$. By direct inspection, our assumptions and identifications are consistent, when we make the legitimate smallness assumption $\mathbf{b} < 10^{-3}$, that ensures that a, $\mathfrak{A}$ satisfy **(S1)**. This condition is subsumed in (8.37) below.

Convention 8.6. For the entire proof, $\mathbf{c}_3(\,\cdot\,)$ and $\mathbf{c}_4(\,\cdot\,)$, as well as $\mathbf{c}_6(\,\cdot\,)$ and $\mathbf{c}_7(\,\cdot\,)$ are defined as in Proposition 7.7 (Refined Energy Estimate) and Proposition 8.3. Furthermore, $\mathbf{c}_8(R) > 1$ will always denote a constant, such that the Sobolev inequality

$$\textbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) \;\leq \mathbf{c}_8(R)\sqrt{E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)} \tag{8.35}$$

holds for all $(\xi_0, b, t) \in \mathbb{R}^2 \times [1,2] \times (-\infty, -\mathfrak{d}^{-1})$ and all vector valued C^R functions f. See (7.43) for the Sobolev inequality and (7.40) for the definition of $\mathfrak{d}$.

The smallness condition on $\mathbf{b}$. Set $J_0 = \epsilon$, with ϵ as in Theorem 8.1, and

$$\begin{aligned} X &= \big(R, J_0, |\mathcal{Q}_1|, |\mathcal{Q}_2|, |\mathcal{Q}_3|\big) \\ X^* &= \big(\,0, J_0, |\mathcal{Q}_1|, |\mathcal{Q}_2|, |\mathcal{Q}_3|\big) \\ \widehat{X} &= \big(\,0, J_0, |\widehat{\mathcal{Q}}_1|, |\widehat{\mathcal{Q}}_2|, |\widehat{\mathcal{Q}}_3|\big) \end{aligned}$$

where $\mathcal{Q}_i$ and $\widehat{\mathcal{Q}}_i$ are fixed as in **(S7)** and $\widehat{\textbf{(S4)}}$, respectively. Also set

$$\mathbf{c}_2'(B) = \tfrac{1}{2}\,\min\big\{\mathbf{c}_3(X),\; \mathbf{c}_3(X^*),\; \mathbf{c}_3(\widehat{X})\big\} \in (0,1) \tag{8.36}$$

The right hand side of (8.36) only depends on B, which justifies the notation $\mathbf{c}_2'(B)$. We impose the legitimate smallness condition

$$\mathbf{b} \;<\; \min\big\{10^{-3},\; \mathbf{c}_6(R)\,\mathbf{c}_2'(B)\big\} \tag{8.37}$$

It is the only smallness condition on $\mathbf{b}$ in the entire proof.

The first smallness condition on $\mathbf{c}$. Set

$$
\begin{aligned}
&Y = \big(R, K, \max_{\sigma\in\{-,+\}} \|\textbf{DATA}^{\sigma}\|_{C^{R+2K+6}(\mathcal{Q})}\big) \\
&\mathbf{T}(C) = -1 - \max\left\{ \frac{1}{\mathfrak{d}},\ \frac{1}{\mathbf{c}_7(Y)\mathbf{c}_2'(B)},\ \left(\frac{4\,\mathbf{c}_8(R)\,(\mathbf{c}_4(X)+1)}{\mathbf{c}_2'(B)\mathbf{c}_6(R)\mathbf{c}_7(Y)}\right)^{\frac{1}{K+1}} \right\}
\end{aligned}
\tag{8.38}
$$

Observe that $\mathcal{C}(a,\mathfrak{A},2) = \mathcal{Q}$ where $\mathcal{Q}$ is defined in **(S9)**. Therefore, Y depends only on C, and so does the right hand side of (8.38), justifying the notation $\mathbf{T}(C)$. We impose the legitimate smallness condition

$$
\mathbf{c} < \frac{1}{|\mathbf{T}(C)|+2} \tag{8.39}
$$

There will be one more smallness condition on $\mathbf{c}$, later in the proof.

Convention 8.7. The system (8.3a) corresponding to **DATA**$^{\sigma}$ will be denoted (8.3a)$^{\sigma}$. We sometimes suppress the superscript σ and simply write **DATA** and (8.3a), in which case the discussion applies equally to **DATA**$^{\sigma}$ and (8.3a)$^{\sigma}$ for $\sigma = -, +$.

Convention 8.8. In every application of Proposition 8.3, the $\mathbf{c}_2'$ of Proposition 8.3 will be the $\mathbf{c}_2'(B)$ of (8.36).

Remark 8.6. We have $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \big(\mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi\big) = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\left(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Psi\right)$ since the transformation $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is linear, maps $\mathcal{M}_{a,\mathfrak{A}}$ to itself and commutes with the matrix u^{-M}. In particular, if Ψ solves (5.7a), then $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Psi$ solves (5.7a).
Observe that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Psi_K^{\sigma} = \Psi_K^{-\sigma}$, because **DATA**$^{\sigma}$ are Pole-Flip compatible, by assumption. Therefore, $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \big(\Psi_K^{\sigma} + \pi\,\Xi\big) = \Psi_K^{-\sigma} + \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot (\pi\,\Xi)$. In particular, if Ξ solves (8.3a)$^{\sigma}$, then $\pi^{-1}\,\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot (\pi\Xi)$ solves (8.3a)$^{-\sigma}$.

Convention 8.9. For the rest of this proof, we consciously abuse notation and write $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi$ for $\pi^{-1}\,\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot (\pi\Xi)$. See, **(S3)** for the definition of π.

Guide. The proof now proceeds through a sequence of 10 steps. Each step begins with a statement (in italics), that is then proven.

Step 1. The assumptions of Proposition 8.3 up to and including (8.17) *are satisfied for all* $T \le \mathbf{T}(C)$*, if* **DATA** $=$ **DATA**$^{\sigma}$ *for* $\sigma = -, +$.
By direct inspection. Recall that $\mathbf{c}_7(\,\cdot\,)$ is non-increasing in all its arguments.

Step 2. For each $t_0 < \mathbf{T}(C)$ *there is a* $t_1(t_0) \in (t_0, \mathbf{T}(C)]$ *and a smooth solution*

$$
\Xi_{t_0} : [t_0, t_1(t_0)) \times \mathbb{R}^3 \;\to\; \pi^{-1}\mathcal{R} \;\cong\; \mathbb{R}^{31}
$$

to the system $\mathbf{M}(q,\Xi)\Xi = h(q,\Xi)$*, with trivial initial data,* $\Xi_{t_0}(t_0,\,\cdot\,) = 0$*, vanishing identically for* $q^3 < \frac{1}{2}$*, so that* $t_1(t_0) \neq \mathbf{T}(C)$ *implies either one or both of* **(Break)**$_1$ *or* **(Break)**$_2$ *(see Proposition 7.3).*
Apply Parts 1 and 2 of Proposition 7.3, in the context of Part 1 of Proposition 8.3, with $T = \mathbf{T}(C)$.

Remark 8.7. Ξ_{t_0} is a 1-parameter family of solutions, parametrized by $t_0 < \mathbf{T}(C)$.

Step 3. $t_1(t_0) = \mathbf{T}(C)$*, for all* $t_0 < \mathbf{T}(C)$*.*
For each $\xi_0 \in \mathbb{R}^2$, we introduce the set

$$\mathcal{J}(\xi_0, t_0) = \Big\{ t \in [t_0, t_1(t_0)) \;\Big|\; \sup_{\tau\in[t_0,t]} E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\Xi_{t_0}\}(\tau) \leq \big(\mathbf{c}_6(R)\mathbf{c}_2'(B)\big)^2 \Big\}$$

It is an interval and closed as a subset of $[t_0, t_1(t_0))$. By Part 2 of Proposition 8.3 and by (8.38),

$$\sqrt{E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\Xi_{t_0}\}(t_0)} \leq \frac{1}{\mathbf{c}_7(Y)\,|\mathbf{T}(C)|^{K+1}} \leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{4}.$$

Therefore, $t_0 \in \mathcal{J}(\xi_0, t_0)$. By continuity of the energy, $\mathcal{J}(\xi_0, t_0)$ contains at least one point different from t_0. For every $t^* \in \mathcal{J}(\xi_0, t_0)$, $t^* > t_0$, the assumptions of Proposition 8.3, Part 3, **(Sys1)** are satisfied with $T = t^*$. It follows from Conclusion 1 that

$$\Xi_{t_0}(q) \in \overline{B_{1/2}(0)} \subset \mathbb{R}^{31} \qquad \text{for all} \qquad q \in \textstyle\bigcup_{t\in(t_0,t^*)}\{t\} \times \mathcal{O}(\xi_0, 2, t) \tag{8.40}$$

By Conclusion 2 and

$$K + 1 \geq J_0, \qquad \mathbf{c}_2'(B) \leq \mathbf{c}_3(X), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}_2'(B) \leq -1/\mathbf{c}_3(X)$$

we can apply the Refined Energy Estimate (7.42) in Proposition 7.7 in the context of **(Sys1)**. Combining (7.42) with Part 2 of Proposition 8.3, one obtains

$$\begin{aligned}\sqrt{E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\Xi_{t_0}\}(\tau)} \leq \frac{2\mathbf{c}_4(X)}{\mathbf{c}_7(Y)\,|\tau|^{K+1}} &\leq \frac{2\,\mathbf{c}_4(X)}{\mathbf{c}_7(Y)\,|\mathbf{T}(C)|^{K+1}} \\ &\leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{2\,\mathbf{c}_8(R)} \leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{2}\end{aligned} \tag{8.41}$$

for all $\tau \in (t_0, t^*)$. The second inequality is self-evident. For the third, use (8.38) again.

The continuity of the energy $E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\Xi_{t_0}\}(\tau)$ for $\tau \in [t_0, t_1(t_0))$ implies that (8.41) holds for $\tau = t^*$, and, consequently, for all $\tau \in \mathcal{J}(\xi_0, t_0)$. It follows that $\mathcal{J}(\xi_0, t_0)$ is also open as a subset of $[t_0, t_1(t_0))$. The set $\mathcal{J}(\xi_0, t_0)$ is nonempty, open and closed as a subset of $[t_0, t_1(t_0))$, and we conclude $\mathcal{J}(\xi_0, t_0) = [t_0, t_1(t_0))$. The upshot is that (8.40) holds with t^* replaced by $t_1(t_0)$, for all $\xi_0 \in \mathbb{R}^2$, and therefore $\Xi_{t_0}\big([t_0, t_1(t_0)) \times \mathcal{Q}\big) \subset \overline{B_{1/2}(0)} \subset \mathbb{R}^{31}$ because

$$[t_0, t_1(t_0)) \times \mathcal{Q} \;\subset\; [t_0, t_1(t_0)) \times \mathbb{R}^2 \times (0, 2) \;=\; \bigcup_{\xi_0\in\mathbb{R}^2}\;\bigcup_{t\in[t_0,t_1(t_0))} \{t\} \times \mathcal{O}(\xi_0, 2, t). \tag{8.42}$$

Now, $\textbf{(Break)}_1$ (see Step 2) is excluded. The inclusion (8.42), the fact that $\mathcal{J}(\xi_0, t_0) = [t_0, t_1(t_0))$ and the Sobolev inequality, (8.35), exclude $\textbf{(Break)}_2$. By Step 2, we conclude that $t_1(t_0) = \mathbf{T}(C)$, for all $t_0 < \mathbf{T}(C)$.

Remark 8.8. A byproduct of the proof of Step 3 is:

$$\text{The inequality (8.41) holds for all } \tau \in [t_0, \mathbf{T}(C)) \text{ and } \xi_0 \in \mathbb{R}^2. \tag{8.43}$$

Convention 8.10 (for Steps 4 and 5). We introduce a new field that is used in the next two steps. It is the restriction of Ξ_{t_0} to $[t_0, \mathbf{T}(C)) \times \mathcal{W}$, where

$$\mathcal{W} \,=\, D_{\frac{5}{2}|\frac{a}{\mathfrak{A}}|}(0) \times (0,1) \,\subset\, \mathbb{R}^3$$

Consciously abusing notation, we will denote this new field by the same symbol Ξ_{t_0}. The new field Ξ_{t_0} is smooth and extends, with all its derivatives, continuously to $[t_0, \mathbf{T}(C)) \times \overline{\mathcal{W}} \subset \mathbb{R}^4$.

Step 4. Ξ_{t_0} is a solution to (8.3a)*, for each $t_0 < \mathbf{T}(C)$.*
Define the open cover $(\mathcal{W}_1, \mathcal{W}_2)$ of $\mathcal{W}$,

$$\mathcal{W}_1 = \mathcal{W} \cap \left(\mathbb{R}^2 \times (0, \tfrac{1}{2})\right) \qquad\qquad \mathcal{W}_2 = \mathcal{W} \cap \left(\mathbb{R}^2 \times (\tfrac{1}{3}, 1)\right)$$

If $\mathbf{q} \in \mathcal{W}_1$, then $\Xi_{t_0}(q) = 0$ (see, Step 2) and **Src**$(q) = 0$ (see, **(S2)** and **(S4)**). In this case, Ξ_{t_0} is self-evidently a solution to (8.3a). On the other hand, if $\mathbf{q} \in \mathcal{W}_2$, then $\psi(q) = 1$ (see, **(S9)**). By direct inspection of **(S10)**, the equation $\mathbf{M}(q, \Xi)\Xi = h(q, \Xi)$ collapses to (8.3a).

Remark 8.9. Recall from Remark 7.3 that, for all $(\xi_0, b, t) \in \mathbb{R}^2 \times [1,2] \times (-\infty, -\frac{1}{\mathfrak{d}})$, the set $\mathcal{O}(\xi_0, b, t)$ is a bundle over the $\underline{u}$-interval $(0, b)$ whose fibers are disks centered at ξ_0, with radii $< \frac{1}{2}$.

Step 5. For each $t_0 < \mathbf{T}(C)$, let $\Xi^{\sigma}_{t_0}$ be the solution to (8.3a)$^{\sigma}$*. Let*

$$\begin{aligned}
\mathcal{Y}(t_0) &= [t_0, \mathbf{T}(C)) \times \left\{ \tfrac{2}{5}|\tfrac{a}{\mathfrak{A}}| < |\xi| < \tfrac{5}{2}|\tfrac{a}{\mathfrak{A}}| \right\} \times (0,1) \\
\mathcal{Z}(t_0) &= \bigcup_{\tau \in [t_0, \mathbf{T}(C))} \;\; \bigcup_{|\frac{a}{\mathfrak{A}}| \le |\xi_0| < 2|\frac{a}{\mathfrak{A}}|} \{\tau\} \times \mathcal{O}(\xi_0, 1, \tau) \;\;\subset\;\; \mathcal{Y}(t_0)
\end{aligned}$$

Note that both $\Xi^{\sigma}_{t_0}$ and $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$ (see, Convention 8.9) are defined on $\mathcal{Y}(t_0)$. We claim that $\Xi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$ on $\mathcal{Z}(t_0) \subset \mathcal{Y}(t_0)$.
The argument is by finite speed of propagation. For any $|\frac{a}{\mathfrak{A}}| \le |\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, let

$$\begin{aligned}
&\mathcal{I}(\xi_0, t_0) = \\
&\left\{ t \in [t_0, \mathbf{T}(C)) \;\middle|\; \Xi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0} \quad \text{on} \quad \bigcup_{\tau \in [t_0, t]} \{\tau\} \times \mathcal{O}(\xi_0, 1, \tau) \;\subset\; \mathcal{Z}(t_0) \right\}
\end{aligned}$$

Our goal is $\mathcal{I}(\xi_0, t_0) = [t_0, \mathbf{T}(C))$, for each $|\frac{a}{\mathfrak{A}}| \le |\xi_0| < 2|\frac{a}{\mathfrak{A}}|$. First of all, $\mathcal{I}(\xi_0, t_0)$ is an interval that contains t_0, because $\Xi^{\sigma}_{t_0}$ and $\Xi^{-\sigma}_{t_0}$ have trivial initial data. By continuity, it is closed as a subset of $[t_0, \mathbf{T}(C))$. We now show that $\mathcal{I}(\xi_0, t_0)$ is also open as a subset of $[t_0, \mathbf{T}(C))$.
By the second to last inequality in (8.43), and by (8.35),

$$\sup_{\tau \in [t_0, \mathbf{T}(C))} \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0, 1, \tau)} \{\Xi^{\sigma}_{t_0}\}(\tau) \;\le\; \tfrac{1}{2}\, \mathbf{c}_6(R) \mathbf{c}'_2(B).$$

Let $t' \in \mathcal{I}(\xi_0, t_0)$. By the definition of $\mathcal{I}(\xi_0, t_0)$, and by continuity

$$\sup_{\tau \in [t_0, t^*)} \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0, 1, \tau)} \{\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}\}(\tau) \;\le\; \tfrac{3}{4}\, \mathbf{c}_6(R) \mathbf{c}'_2(B)$$

for some $t^* \in (t', \mathbf{T}(C))$. The assumptions of Proposition 8.3, Part 3, **(Sys2)** are satisfied with $T = t^*$, $\varXi^{(1)} = \varXi^{\sigma}_{t_0}$, $\varXi^{(2)} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \varXi^{-\sigma}_{t_0}$. Conclusion 2 of Proposition 8.3 enables us to apply Proposition 7.7 (Refined Energy Estimate) for **(Sys2)**, with $J = \frac{1}{2}$. The assumptions of Proposition 7.7 are satisfied because

$$J = \tfrac{1}{2} \geq J_0, \qquad \mathbf{c}_2'(B) \leq \mathbf{c}_3(X^*), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}_2'(B) \leq -1/\mathbf{c}_3(X^*). \tag{8.44}$$

In the present case, the Refined Energy Estimate (7.42) becomes

$$\sqrt{E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\varUpsilon\}(\tau)} \leq \mathbf{c}_4(X^*) \, \frac{|t_0|^{1/2} \sqrt{E^0_{\mathcal{O}(\xi_0,1,t_0)}\{\varUpsilon\}(t_0)} + 0}{|\tau|^{1/2}}$$

for all $\tau \in (t_0, t^*)$, where $\varUpsilon = \varXi^{\sigma}_{t_0} - \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \varXi^{-\sigma}_{t_0}$. Furthermore, the energy on the right hand side is zero. The vanishing of the energy on the left hand side implies $[t_0, t^*) \subset \mathcal{I}(\xi_0, t_0)$. Hence, $\mathcal{I}(\xi_0, t_0)$ is an open subset of $[t_0, \mathbf{T}(C))$.

Convention 8.11 (for the remaining steps). We introduce a new field for the remaining steps. For each $t_0 < \mathbf{T}(C)$ and $\sigma \in \{-,+\}$, it is the map (see, Convention 8.9)

$$\begin{aligned} [t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \;&\to\; \pi^{-1}\mathcal{R} \;\cong\; \mathbb{R}^{31} \\ q = (t, \xi, \underline{u}) \;&\mapsto\; \begin{cases} \varXi^{\sigma}_{t_0}(q) & \text{if } |\xi| < 2|\frac{a}{\mathfrak{A}}| \\ \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \varXi^{-\sigma}_{t_0}(q) & \text{if } |\xi| > \frac{1}{2}|\frac{a}{\mathfrak{A}}| \end{cases} \end{aligned} \tag{8.45}$$

It is well defined on the flip-invariant $[t_0, \mathbf{T}(C)) \times \left\{\frac{1}{2}|\frac{a}{\mathfrak{A}}| < |\xi| < 2|\frac{a}{\mathfrak{A}}|\right\} \times (0,1)$, which is contained in $\mathcal{Z}(t_0) \cup \left(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{Z}(t_0)\right)$ by Step 5. It coincides with $\varXi^{\sigma}_{t_0}$ on the set $\mathcal{Z}(t_0)$ of Step 5. Consciously abusing notation, we will denote this new field by the same symbol $\varXi^{\sigma}_{t_0}$.

Step 6. For each $t_0 < \mathbf{T}(C)$ and $\sigma \in \{-,+\}$,

$$\varXi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \varXi^{-\sigma}_{t_0} \qquad \textit{on} \qquad [t_0, \mathbf{T}(C)) \times (\mathbb{R}^2 \setminus \{0\}) \times (0,1)$$

The field $\varXi^{\sigma}_{t_0} : [t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}$ is smooth, vanishes when $q^3 < \frac{1}{2}$, and extends, with its derivatives of all orders, continuously to $[t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$. Moreover, it is a solution to (8.3a)$^{\sigma}$ *on its entire domain of definition and satisfies*

$$\sup_{|\xi_0| < 2|\frac{a}{\mathfrak{A}}|} \; \sup_{\tau \in (t_0, \mathbf{T}(C))} \mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,1,\tau)}\{\varXi^{\sigma}_{t_0}\}(\tau) \leq \tfrac{1}{2}\, \mathbf{c}_6(R)\mathbf{c}_2'(B) \tag{8.46a}$$

$$\sup_{|\xi_0| < 2|\frac{a}{\mathfrak{A}}|} \; \sup_{\tau \in (t_0, \mathbf{T}(C))} |\tau|^{K+1}\, \mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,1,\tau)}\{\varXi^{\sigma}_{t_0}\}(\tau) \leq \frac{2\mathbf{c}_4(X)\mathbf{c}_8(R)}{\mathbf{c}_7(Y)} \tag{8.46b}$$

The main point before (8.46a) is that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is a field configuration symmetry. The inequalities (8.46a), (8.46b) are consequences of (8.43) and the Sobolev inequality (8.35). Be aware that that the $\varXi_{t_0}$ in (8.43) is related to the present $\varXi^{\sigma}_{t_0}$ by (8.45).

Step 7. For each $t_0 < \mathbf{T}(C) - 1$ and $\sigma \in \{-,+\}$,

$$\sup_{|\xi_0| < 2|\frac{a}{\mathfrak{A}}|} \; \sup_{\tau \in (t_0+1, \mathbf{T}(C))} E^0_{\mathcal{O}(\xi_0,1,\tau)}\left\{\left(\varXi^{\sigma}_{t_0}\right)^{\sharp}\right\}(\tau) \;\lesssim_{(Y,J_0)}\; \frac{1}{|t_0|}.$$

where $\big(\Xi^\sigma_{t_0}\big)^\sharp$ *is the constraint field associated to the field* (8.45).
For any $|\xi_0| < 2\,|\frac{a}{\mathfrak{A}}|$ and any $t^* \in (t_0+1, \mathbf{T}(C))$, the assumptions of Proposition 8.3, Part 3, **(Sys3)** are satisfied with $T = t^*$. By Conclusion 3, $R \geq 4,\ K \geq 0$, and (8.46b),

$$\sup_{\tau \in (t_0+1, t^*)} |\tau|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,\tau)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(\tau) \ \lesssim_{(Y,J_0)}\ 1.$$

By continuity, this holds for $t = t_0 + 1$ as well. Therefore, the energy satisfies

$$|t_0+1|^2\, E^0_{\mathcal{O}(\xi_0,1,t_0+1)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(t_0+1) \ \lesssim_{(Y,J_0)}\ 1. \tag{8.47}$$

By Conclusion 2, we can apply Proposition 7.7 (Refined Energy Estimate) for **(Sys3)**, with $J = \frac{1}{2},\ \mathcal{I} = (t_0+1, t^*)$. The assumptions of Proposition 7.7 are satisfied, because

$$J = \tfrac{1}{2} \geq J_0, \qquad \mathbf{c}_2'(B) \leq \mathbf{c}_3(\widehat{X}), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}_2'(B) \leq -1/\mathbf{c}_3(\widehat{X}).$$

The Refined Energy Estimate (7.42) and (8.47) imply

$$\sup_{\tau \in (t_0+1, t^*)} E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(\tau) \ \lesssim_{(Y,J_0)}\ \frac{1}{|t_0+1|} \ \lesssim_{(Y,J_0)}\ \frac{1}{|t_0|}.$$

Step 8. For all $\sigma \in \{-,+\}$ *and all* $t_1 \leq t_2 < \mathbf{T}(C)$,

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau \in (t_2, \mathbf{T}(C))} E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_2} - \Xi^\sigma_{t_1}\}(\tau) \ \lesssim_{(Y,J_0)}\ \frac{1}{|t_2|}$$

For any $t^* \in (t_2, \mathbf{T}(C))$ and $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, the assumptions in Proposition 8.3, Part 3, for **(Sys2)**, are satisfied with $t_0 = t_2$, $T = t^*$ and $\Xi^{(1)} = \Xi^\sigma_{t_1}$, $\Xi^{(2)} = \Xi^\sigma_{t_2}$. By Conclusion 2, we can apply Proposition 7.7 (Refined Energy Estimate) with $J = \frac{1}{2}$, see (8.44). The Refined Energy Estimate (7.42) implies

$$E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_2} - \Xi^\sigma_{t_1}\}(\tau) \ \leq\ \big(\mathbf{c}_4(X^*)\big)^2\, \frac{|t_2|\, E^0_{\mathcal{O}(\xi_0,1,t_2)}\{\Xi^\sigma_{t_1}\}(t_2)}{|\tau|} \ \lesssim_{(Y,J_0)}\ \frac{1}{|t_2|^{2K+1}}$$

for all $\tau \in (t_2, t^*)$. For the second inequality, see (8.46b).

Step 9. There exist

$$\Xi^\sigma : (-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}, \qquad \sigma \in \{-,+\}$$

with $\Xi^\sigma = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}$ *(see, Convention 8.9), that vanish when* $q^3 < \frac{1}{2}$*, are* C^{R-3}*, and extend, with their derivatives of all orders* $\leq R-3$*, continuously to* $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$*. Moreover, they are solutions to both* (8.3a)$^\sigma$ *and* $(\Xi^\sigma)^\sharp = 0$*, and*

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau \in (-\infty, \mathbf{T}(C))} \mathbf{Sup}^{(R-3)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma\}(\tau) \ \leq\ \tfrac{1}{2}\, \mathbf{c}_6(R)\mathbf{c}_2'(B) \tag{8.48a}$$

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau \in (-\infty, \mathbf{T}(C))} |\tau|^{K+1}\, \mathbf{Sup}^{(R-3)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma\}(\tau) \ \lesssim_{(Y,J_0)}\ 1 \tag{8.48b}$$

$$\sup_{\tau \in (-\infty, \mathbf{T}(C))} |\tau|^{K+1}\, \mathbf{Sup}^{(R-3)}_{D_{4|\frac{a}{\mathfrak{A}}|}(0)\times(0,1)}\{\Xi^\sigma\}(\tau) \ \lesssim_{(Y,J_0)}\ 1 \tag{8.48c}$$

$$\tfrac{1}{2} \leq \mathbf{B}^0(q, \Xi(q)) \leq 2 \text{ for all } q \in (-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \tag{8.48d}$$

For each $\beta \in (0,1)$, introduce the compact set

$$\mathcal{X}_\beta = [\mathbf{T}(C) - \beta^{-1}, \mathbf{T}(C) - \beta] \times \overline{D_{2|\frac{a}{\mathfrak{A}}|}(0)} \times [0,1]$$

For every sequence $t_n \to -\infty$, with $t_n < \mathbf{T}(C) - \beta^{-1}$, the sequence of fields $\Xi^\sigma_{t_n}$ is, by Step 8, a Cauchy sequence in $L^2(\mathcal{X}_\beta)$. Set $\Xi^\sigma|_{\mathcal{X}_\beta} = L^2\text{-}\lim_{t\to-\infty} \Xi^\sigma_t$. By (8.46a), the 1-parameter family Ξ^σ_t with $t < \mathbf{T}(C) - \beta^{-1}$ is a bounded subset of $C^{R-2}(\mathcal{X}_\beta)$, the space of $\pi^{-1}\mathcal{R} \cong \mathbb{R}^{31}$ valued functions of class C^{R-2} on the interior of $\mathcal{X}_\beta$, that extend continuously, with their derivatives of all orders $\leq R-2$, to the boundary. By Arzela-Ascoli, there is a subsequence that converges in $C^{R-3}(\mathcal{X}_\beta)$. Therefore, $\Xi^\sigma|_{\mathcal{X}_\beta}$ is in $C^{R-3}(\mathcal{X}_\beta)$. It follows that Ξ^σ is C^{R-3} on the interior of $\bigcup_{\beta\in(0,1)} \mathcal{X}_\beta$, and extends with its derivatives of all orders $\leq R-3$ continuously to $\bigcup_{\beta\in(0,1)} \mathcal{X}_\beta = (-\infty, \mathbf{T}(C)) \times \overline{D_{2|\frac{a}{\mathfrak{A}}|}(0)} \times [0,1]$. By construction, $\Xi^\sigma = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}$ on $\mathcal{X}_\beta \cap (\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{X}_\beta)$. Hence, the pair of fields Ξ^σ have unique C^{R-3} Pole-Flip compatible extensions to $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1)$, which extend with their derivatives of all orders $\leq R-3$ continuously to $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$, as required by Step 9.
It follows directly from Step 6, that the pair Ξ^σ has all the desired properties, including the bounds (8.48a), (8.48b) (recall that $R-3 \geq 1$), with the exception of $(\Xi^\sigma)^\sharp = 0$, (8.48c) and (8.48d). It is implicit in our construction that for each $\beta \in (0,1)$, there is a sequence $t_n \to -\infty$ so that $\Xi^\sigma_{t_n} \to \Xi^\sigma$ in $C^1(\mathcal{X}_\beta)$, and therefore $(\Xi^\sigma_{t_n})^\sharp \to (\Xi^\sigma)^\sharp$ in $C^0(\mathcal{X}_\beta)$. Now, by step 7, $(\Xi^\sigma)^\sharp|_{\mathcal{X}_\beta} = 0$, for all $\beta \in (0,1)$. By Pole-Flip compatibility, $(\Xi^\sigma)^\sharp = 0$ everywhere. The estimate (8.48c) follows from (8.48b) when $|\xi| < 2|\frac{a}{\mathfrak{A}}|$. For $\frac{1}{2}|\frac{a}{\mathfrak{A}}| < |\xi| < 4|\frac{a}{\mathfrak{A}}|$, it also follows from (8.48b), by using Pole-Flip compatibility and Lemma F.1 in Appendix F. To verify (8.48d), observe that the assumptions of Proposition 8.3, Part 3, for **(Sys2)**, are satisfied for any $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, $t_0 < T < \mathbf{T}(C)$, $\Xi^{(1)} = \Xi^{(2)} = \Xi^\sigma$. By Conclusion 2, **(RE1)** holds in the context of **(Sys2)**, which implies (8.48d) for $q \in (-\infty, \mathbf{T}(C)) \times D_{2|\frac{a}{\mathfrak{A}}|}(0) \times (0,1)$, and for general q by Pole-Flip compatibility.

Step 10. The fields Ξ^σ in Step 9 are unique in the following sense: Suppose, for some $t_1 < \mathbf{T}(C)$, the C^1-fields $\widetilde{\Xi}^\sigma : (-\infty, t_1) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}$ are Pole-Flip compatible, extend with their first derivatives continuously to $(-\infty, t_1] \times \mathbb{R}^2 \times [0,1]$, are solutions to (8.3a)$^\sigma$, vanish when $q^3 < \frac{1}{2}$, and satisfy

$$\lim_{\tau\to-\infty} \sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|} |\tau|^{J_0} \,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,\tau)}\{\widetilde{\Xi}^\sigma\}(\tau) \,=\, 0 \tag{8.49}$$

Then, $\widetilde{\Xi}^\sigma = \Xi^\sigma$ on $(-\infty, t_1) \times \mathbb{R}^2 \times (0,1)$.
For every sufficiently negative $\tau < t_1$, the assumptions of Proposition 8.3, Part 3, for **(Sys2)**, are satisfied for any $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, $t_0 < T = \tau$, $\Xi^{(1)} = \Xi^\sigma$, $\Xi^{(2)} = \widetilde{\Xi}^\sigma$. The condition that τ is sufficiently negative is used to verify hypothesis (iv) of Part 3 of Proposition 8.3. By Conclusion 2, we can apply (checking, the additional hypothesis similarly to (8.44)) Proposition 7.7 with $J = J_0$. It follows from (7.42) that

$$E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(\tau) \;\leq\; \big(\mathbf{c}_4(X^*)\big)^2\, \frac{|t_0|^{2J_0}\, E^0_{\mathcal{O}(\xi_0,1,t_0)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(t_0)}{|\tau|^{2J_0}}.$$

We take the limit $t_0 \to -\infty$, keeping τ fixed. By (8.48b) and (8.49), and the fact that $2J_0 < 1$, we conclude that the energy $E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(\tau) = 0$. Exploiting

the Pole-Flip compatibility, $\varXi^\sigma(\tau,\,\cdot\,) = \widetilde{\varXi}^\sigma(\tau,\,\cdot\,)$ for all sufficiently negative τ. To demonstrate that $\varXi^\sigma = \widetilde{\varXi}^\sigma$ on $(-\infty, t_1)\times\mathbb{R}^2\times(0,1)$, we make a closed-open argument almost identical to the one in the proof of Step 5.

We finally return from the $q = (t, \xi, \underline{u})$ to the $x = (\xi, \underline{u}, u)$ coordinate system, and complete the proof of Theorem 8.1. The x-set **Strip**$(1, \mathbf{c})$ is contained in the q-set $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1)$, by the smallness condition (8.39).

Existence in Theorem 8.1 follows from Step 9, with $\varPsi^\sigma = \varPsi^\sigma_K + \pi\,\varXi^\sigma$. We only have to check (8.33), (8.34) and conditions $(\star)$ and $(\star\star)$ (see, Definitions 2.1 and 2.2), and apply Proposition 2.2. Write

$$\varPsi^\sigma(\,\cdot\,,u) - \varPsi^\sigma(0)(\,\cdot\,,u) = \tfrac{1}{u}\sum_{k=0}^{K}(\tfrac{1}{u})^k\,\varPsi^\sigma(k+1)(\,\cdot\,) + \pi\,\varXi^\sigma(\,\cdot\,,u)$$

$$\varPsi^\sigma(\,\cdot\,,u) - \sum_{k=0}^{K}(\tfrac{1}{u})^k\,\varPsi^\sigma(k)(\,\cdot\,) = \tfrac{1}{u^{K+1}}\,\varPsi^\sigma(K+1)(\,\cdot\,) + \pi\,\varXi^\sigma(\,\cdot\,,u)$$

The coefficient functions $\varPsi^\sigma(k+1)$, appearing on the right hand sides, are estimated using $\|\varPsi^\sigma(k+1)\|_{C^{R+1}(\mathcal{C}(a,\mathfrak{A},2))} \lesssim_Y 1$ (see, the Overall Preliminaries in the proof of Proposition 8.3) and $\varXi^\sigma$ is estimated using (8.48c). Now, (8.33) follows. Also (8.34) follows, with an additional legitimate smallness condition on $\mathbf{c}$ depending only on (Y, J_0). Condition $(\star)$ is a consequence of the inequality $\frac{1}{2} \le e_3 \le 2$ on **Strip**$(1,\mathbf{c})$, which follows from (8.48d). Here, e_3 is a component of $\varPhi^\sigma = (e,\gamma,w) = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\varPsi^\sigma$. Finally, the equation $L\big(e_1\overline{e}_2 - \overline{e}_1 e_2\big) = -2\gamma_2\,\big(e_1\overline{e}_2 - \overline{e}_1 e_2\big)$ (a consequence of the first two lines of (2.4)) implies that $\Im\big(e_1\overline{e}_2\big)$ cannot change sign along the integral curves of L, and therefore $\Im\big(e_1\overline{e}_2\big) < 0$ on **Strip**$(1,\mathbf{c})$ because it is negative when $\underline{u} < \frac{1}{2}$. This implies $(\star\star)$.

Uniqueness in Theorem 8.1 follows from Step 10. $\square$

9. Conclusions

Theorem 8.1 is the central mathematical result of Sections 7 and 8. It establishes the existence and uniqueness of classical vacuum fields, under appropriate smallness conditions. In this section we extract a number of corollaries of Theorem 8.1.

First, we show (Proposition 9.1) that the far field formal power series is an asymptotic expansion for the classical solution of Theorem 8.1. In this sense, the solution is quasi-explicit. Using this result, we can systematically exploit Theorem 8.1 by transferring properties of the formal solution to corresponding properties of the classical solution.

The high point of [Chr] is the demonstration that strongly focused gravitational waves generate trapped spheres. We recover this result (Proposition 9.2). One would like to show that the trapped sphere is just a shadow of a black hole, that is, a horizon forms. We take a first step by showing (Proposition 9.5) that for special initial conditions, the solutions of Theorem 8.1 become arbitrarily close to the Schwarzschild solution on the upper edge of the strip, between the trapped surface and past null infinity. This would be used in conjunction with a controlled perturbation expansion around the Schwarzschild / Kerr family to construct the global exterior of a black hole.

An important feature of our approach is that the scaling limit $a = \mathfrak{A} \downarrow 0$ is regular, see Remark 6.3. Informally, "**(SHS)** plus $\varPsi^\sharp = 0$" (see, (5.7a)) degenerates to a

2-dimensional system, separately for each ξ. Its solutions break down in finite time. We speculate in Subsection 9.5 that an appropriate expansion in $\mathfrak{A}$ around this two dimensional system can be used to explore the spacetime singularity.

We also obtain, in Subsection 9.6, a class of solutions distinct from that of [Chr], corresponding to more general initial data at past null infinity. This class arises by from the limit $\mathfrak{A} \downarrow 0$ with $a > 0$ fixed.

We assume, without further comment, the definitions and conventions of Section 2 through Section 6 and Theorem 8.1. However, see the Index of Notation, Appendix A.

Proposition 9.1 (Asymptotic expansion). *Let $\Phi^\sigma = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^\sigma$ be the pair of Pole-Flip compatible vacuum fields of Theorem 8.1 for $K = 0$. For each $L \geq 0$,*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^4 \\ |\alpha| \leq R-3}} \Big\| \partial^\alpha \Big(\Psi^\sigma(\,\cdot\,, u) - \sum_{k=0}^{L} \frac{\Psi^\sigma(k)(\,\cdot\,)}{u^k} \Big) \Big\|_{C^0(\mathcal{C}(a,\mathfrak{A},1))} = \mathcal{O}\Big(\frac{1}{|u|^{L+1}} \Big)$$

as $u \to -\infty$. In other words, the far field formal power series $[\,\Psi^\sigma\,]$ is an asymptotic expansion for Ψ^σ.

Proof. Observe that the conditions imposed on the data a, $\mathfrak{A}$, $\mathbf{DATA}^\sigma$, R, ϵ in Theorem 8.1 are independent of K. Therefore, Theorem 8.1 can be applied with the same data for all $K \geq 0$. For each $K \geq 0$, we obtain a pair of Pole-Flip compatible vacuum fields on $\mathbf{Strip}(1, \lambda_K) \subset \mathbb{R}^4$, where $\lambda_K > 0$ depends on K through the vector C in Theorem 8.1. The vacuum fields corresponding to any pair $K,\ K' \geq 0$ coincide for sufficiently negative u, by the uniqueness statement of Theorem 8.1. In particular, this is true for $K = 0$, $K' = L$. The bound (8.34) for the $K' = L$ vacuum field implies the proposition. $\square$

9.1. Three Points of View. It is helpful to consider the focusing of gravitational waves from three perspectives, that yield three different pictures.

Regularized Picture (R)

$\big\uparrow$ Field transformation $\mathfrak{C} \circ \mathfrak{A}$
where $\mathfrak{C}(\xi) = a\,\xi$

High Amplitude Picture (H)

$\big\downarrow$ Isotropic scaling transformation $\mathfrak{J}$
with scaling constant $\mathfrak{J} = \mathfrak{A}^4$

Finite Mass Picture (F)

First, recall from Section 3 that the Isotropic Scaling $\mathfrak{J}$, the Anisotropic Scaling $\mathfrak{A}$ and the Angular Coordinate Transformation $\mathfrak{C}$ are field symmetries (see, Definition 3.1). Their isotropic respectively anisotropic character refers to their action on the frame and the coordinate system. Both scalings are, at the level of the Lorentzian metric, global conformal transformations.

The pictures are fixed by the table

	Regularized	**High Amplitude**	**Finite Mass**
Background	$\mathcal{M}_{a,\mathfrak{A}}$	$\mathcal{M}_{1,1}$	$\mathcal{M}_{1,1}$
Data	$\eta^\sigma\big(\xi,\,\underline{u}\big)$	$\mathfrak{A}^{-2}\,\eta^\sigma\big(\frac{a}{\mathfrak{A}}\xi,\,\underline{u}\big)$	$\mathfrak{A}^{-2}\,\eta^\sigma\big(\frac{a}{\mathfrak{A}}\xi,\,\mathfrak{A}^{-4}\underline{u}\big)$
Domain	$\textbf{Strip}\big(1,\mathbf{c}\big)$	$\textbf{Strip}\big(1,\mathbf{c}\,\mathfrak{A}^2\big)$	$\textbf{Strip}\big(\mathfrak{A}^4,\mathbf{c}\,\mathfrak{A}^{-2}\big)$
Hemisphere	$\lvert\xi\rvert < \lvert\frac{a}{\mathfrak{A}}\rvert$	$\lvert\xi\rvert < 1$	$\lvert\xi\rvert < 1$

The Regularized Picture is the arena of Theorem 8.1. The other two are obtained, as indicated above, by scaling. Row 1 displays the Minkowski background fields. Row 2 gives the functional form of the initial data at past null infinity. More precisely,

$$\textbf{DATA}^\sigma_{\mathrm{R}}(\xi,\underline{u}) = \eta^\sigma(\xi,\underline{u}) \qquad \Phi^\sigma_R = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^\sigma_R \tag{9.1a}$$

$$\textbf{DATA}^\sigma_{\mathrm{H}}(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^\sigma\big(\tfrac{a}{\mathfrak{A}}\xi,\,\underline{u}\big) \qquad \Phi^\sigma_H = \mathcal{M}_{1,1} + u^{-M}\Psi^\sigma_H \tag{9.1b}$$

$$\textbf{DATA}^\sigma_{\mathrm{F}}(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^\sigma\big(\tfrac{a}{\mathfrak{A}}\xi,\,\mathfrak{A}^{-4}\underline{u}\big) \qquad \Phi^\sigma_F = \mathcal{M}_{1,1} + u^{-M}\Psi^\sigma_F \tag{9.1c}$$

The first equation gives two new names for the $\textbf{DATA}^\sigma(\xi,\underline{u})$ of Theorem 8.1. Row 3 displays the functional dependence of the domains on the scaling parameter $\mathfrak{A}$, with the notation $\textbf{Strip}(\mu,\lambda) = \mathbb{R}^2\times(0,\mu)\times(-\infty,-\lambda^{-1})$. Row 4 gives the size of the ξ-disk which, under stereographic projection, corresponds to one hemisphere of S^2.

9.2. Two Physical Regimes. There are two natural physical regimes. Informally, both appear as limits $\mathfrak{A}\downarrow 0$ in the Regularized Picture, keeping the data η^σ fixed. They are distinguished by:

- *2D (Scaling) Limit*: $a = \mathfrak{A}\downarrow 0$.
- *4D (Scaling) Limit*: a fixed, $\mathfrak{A}\downarrow 0$.

(The 4D Limits breaks Pole-Flip compatibility of η^σ. This will be discussed below.)

Definition 9.1. *The* 2D Limit Assumptions *are the hypothesis, with $a = \mathfrak{A}$ and $\epsilon = \frac{1}{4}$, of Theorem 8.1 on $\mathfrak{A}$, $\textbf{DATA}^\sigma = \eta^\sigma$, R and K up to and including condition* (8.32).

Remark 9.1. Explicitly, the 2D Limit Assumptions are: $R\geq 4,\ K\geq 0,\ \eta^\sigma = 0$ when $\underline{u} < \frac{1}{2}$, and

$$0 < |\mathfrak{A}| < \mathbf{b}, \qquad \eta^\sigma = \textbf{Flip}_1\cdot\eta^{-\sigma}, \qquad \max_{\sigma\in\{-,+\}}\|\eta^\sigma\|_{C^{R+4}(D_4(0)\times(0,2))}\leq\mathbf{b}$$

Here, $\mathbf{b}\in(0,1)$ depends only on R. The constant $\mathbf{c}\in(0,1)$ in Theorem 8.1 depends only on R, K and $\max_{\sigma\in\{-,+\}}\|\eta^\sigma\|_{C^{R+2K+6}(D_4(0)\times(0,2))}$.
The conditions on $\eta^\sigma : \mathbb{R}^2\times(0,\infty)\to\mathbb{C}$ are independent of $\mathfrak{A}$. Also the domain of definition $\textbf{Strip}(1,\mathbf{c})$ of the vacuum field in Theorem 8.1 is independent of $\mathfrak{A}$. Therefore, Theorem 8.1 is consistent with the 2D Limit. The choice $\epsilon = \frac{1}{4}$ is just for concreteness.

Remark 9.2. The intuition behind the designations Regularized Picture and High Amplitude Picture is immediately clear in the context of the 2D Limit. For the Regularized picture, see Remark 6.3. In the High Amplitude Picture, the initial data at past null infinity for the corresponding family of vacuum fields is $\textbf{DATA}^\sigma_H(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^\sigma(\xi,\underline{u})$. It grows unboundedly as $\mathfrak{A}\downarrow 0$. The Finite Mass Picture will be discussed momentarily.

Remark 9.3. From our perspective, [Chr] investigates the 2D Limit ($a = \mathfrak{A}$) in the Finite Mass Picture. Christodoulou's small parameter $\delta > 0$ is to be identified with our $\mathfrak{A}^4$. With this translation, the first equation in (9.1c) is precisely Christodoulou's "short pulse ansatz". For [Chr], the "short pulse hierarchy" plays a central role (see equation (24) on page 20 in [Chr], and the following discussion). In our approach, this hierarchy plays no role at all. However, it can be recovered through the scaling transformations required to go from the Regularized Picture to the Finite Mass Picture, see (9.5c) and (9.6c) below. By contrast, our working picture, the Regularized Picture, merely contains a dichotomy: the $\mathfrak{P}$-even components display one behavior, the $\mathfrak{P}$-odd components another, see Remark 6.3 or (9.8). This dichotomy disappears in our 4D Limit.

9.3. Trapped Spheres.

Proposition 9.2. *Make the 2D Limit Assumptions (Definition 9.1) with $K = 0$. Set*

$$\Lambda(\underline{u}_1) \;=\; \min_{\sigma\in\{-,+\}} \; \inf_{\xi\in D_4(0)} \int_0^{\underline{u}_1} \mathrm{d}\underline{u}\; |\eta^\sigma(\xi,\underline{u})|^2$$

Suppose $\Lambda(\underline{u}_1) > 0$ for some $\underline{u}_1 \in (0,1)$. Then γ_2^σ and γ_6^σ are everywhere negative on the sphere

$$S_{\underline{u},u}: \qquad (\underline{u},u) = \big(\underline{u}_1,\, -\tfrac{1}{2}\,\Lambda(\underline{u}_1)\,\mathfrak{A}^{-2}\big)$$

whenever $\mathfrak{A} \in (0,\mathbf{b})$ is sufficiently small depending only on $\Lambda(\underline{u}_1)$ and $\mathbf{c}$. For instance, $\mathfrak{A} < \frac{1}{4}\sqrt{\mathbf{c}}\,\min\{1, \Lambda(\underline{u}_1)\}$ will do. In other words, $S_{\underline{u},u}$ is a trapped sphere (see, Remark 2.5).

Remark 9.4. Clearly, there is an infinite dimensional family of pairs η^σ, $\mathfrak{A}$ satisfying the assumptions of Proposition 9.2.

Proof. By (8.34), the components γ_2^σ, γ_6^σ of $\Phi^\sigma = \mathcal{M}_{\mathfrak{A},\mathfrak{A}} + u^{-M}\Psi^\sigma$ satisfy:

$$\begin{aligned}
\Big| u^2\gamma_2^\sigma(\xi,\underline{u},u) - \Big(+ \frac{\mathfrak{A}^2u^2}{\mathfrak{A}^2\underline{u} - u} + \omega_2^\sigma(0)(\xi,\underline{u})\Big)\Big| \;&\le\; \frac{1}{\mathbf{c}\,|u|}\\
\Big| u^2\gamma_6^\sigma(\xi,\underline{u},u) - \Big(- \frac{u^2}{\mathfrak{A}^2\underline{u} - u} + \omega_6^\sigma(0)(\xi,\underline{u})\Big)\Big| \;&\le\; \frac{1}{\mathbf{c}\,|u|}
\end{aligned}$$

for all $(\xi,\underline{u},u) \in D_4(0)\times(0,1)\times(-\infty,-\mathbf{c}^{-1})$, where (see, Definition 5.1)

$$\omega_2^\sigma(0)(\xi,\underline{u}) \;=\; -\int_0^{\underline{u}} \mathrm{d}s\,|\eta^\sigma(\xi,s)|^2 \qquad , \qquad \omega_6^\sigma(0)(\xi,\underline{u}) = 0$$

To find a trapped sphere, let $u = -\lambda\mathfrak{A}^{-2}$, where $\lambda > 0$. Now, for all $(\xi,\underline{u}) \in D_4(0)\times(0,1)$ and $\mathbf{c}\lambda > \mathfrak{A}^2$:

$$\begin{aligned}
&\text{If}\quad \lambda - \int_0^{\underline{u}} \mathrm{d}s\,|\eta^\sigma(\xi,s)|^2 + \frac{\mathfrak{A}^2}{\mathbf{c}\lambda} \;<\; 0 \quad , \quad \text{then}\quad \gamma_2^\sigma(\xi,\underline{u},-\lambda\mathfrak{A}^{-2}) \;<\; 0.\\
&\text{If}\quad -\frac{\lambda^2}{\mathfrak{A}^2(\mathfrak{A}^4+\lambda)} + \frac{\mathfrak{A}^2}{\mathbf{c}\lambda} \;<\; 0 \quad , \quad \text{then}\quad \gamma_6^\sigma(\xi,\underline{u},-\lambda\mathfrak{A}^{-2}) \;<\; 0.
\end{aligned} \tag{9.2}$$

Proposition 9.2 is a direct consequence of (9.2) with $\lambda = \frac{1}{2}\Lambda(\underline{u}_1)$, if we also recall that $\mathbf{Flip}_1$ does not change the sign of γ_2^σ and γ_6^σ. $\square$

9.4. The 2D Limit in the Finite Mass Picture for a Finite Duration Pulse. We make the 2D Limit Assumptions (Definition 9.1), and the assumptions for a finite duration pulse:

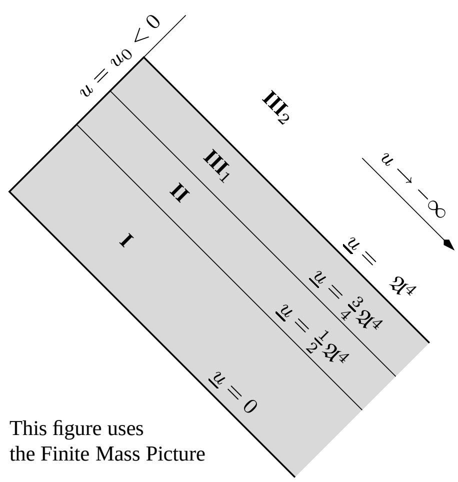
$$\eta^\sigma : \mathbb{R}^2 \times (0,\infty) \to \mathbb{C} \text{ has support contained in } \mathbb{R}^2 \times (\tfrac{1}{2}, \tfrac{3}{4}). \tag{9.3a}$$

$$\int_{1/2}^{3/4} \mathrm{d}\underline{u}\, \eta^\sigma(\xi, \underline{u}) = 0 \text{ for all } \xi \text{ and } \sigma. \tag{9.3b}$$

Remark 9.5. Clearly, there is an infinite dimensional family of Pole-Flip compatible η^σ satisfying (9.3). The conditions (9.3) are equivalent to the condition that $\eta^\sigma(\xi,\underline{u}) = \frac{\partial}{\partial \underline{u}} \mathbf{G}^\sigma(\xi, \underline{u})$ for a function $\mathbf{G}^\sigma : \mathbb{R}^2 \times (0,\infty) \to \mathbb{C}$ with $\operatorname{supp} \mathbf{G}^\sigma \subset \mathbb{R}^2 \times (\frac{1}{2}, \frac{3}{4})$.

Remark 9.6. One can relax the conditions (9.3) by imposing suitable smallness conditions for η^σ on $\mathbb{R}^2 \times (\frac{3}{4}, \infty)$ and for $\int_{1/2}^{3/4} \mathrm{d}\underline{u}\, \eta^\sigma(\xi, \underline{u})$, depending on $\mathfrak{A}$. In this case, contrary to our current convention, η^σ itself would have to be allowed to depend on $\mathfrak{A}$, say polynomially. A smallness condition depending on $\mathfrak{A}$ is an "open" condition, as opposed to (9.3).

We begin with an informal discussion of the 2D Limit $a = \mathfrak{A} \downarrow 0$ in the Finite Mass Picture. Fix $u_0 < 0$. The shaded region in the figure (just below) is $\textbf{Strip}(\mathfrak{A}^4, |u_0|^{-1})$. For $|\mathfrak{A}|$ small enough, it is contained in $\textbf{Strip}(\mathfrak{A}^4, \mathbf{c}\mathfrak{A}^{-2})$, the domain, in the Finite Mass Picture, on which the solution to Theorem 8.1 exists. Region **I** is Minkowski space $\mathcal{M}_{1,1}$. In the "pulse region" **II**, the solution grows unboundedly as $\mathfrak{A} \downarrow 0$. Nevertheless, as we will show, (9.3) implies that on each compact $K \subset \textbf{Strip}(\infty, |u_0|^{-1})$ the far field formal solution converges as $\mathfrak{A} \downarrow 0$. Note that as $\mathfrak{A} \downarrow 0$, the compact K is eventually contained in the "after the pulse region" $\mathbf{III} = \mathbf{III}_1 \cup \mathbf{III}_2$. One would like to have an analogous result for classical solutions. In this paper we take a step towards such a result, by controlling the solution in $\mathbf{III}_1$, a strip moving and shrinking with $\mathfrak{A}$.



If $\int_{1/2}^{3/4} \mathrm{d}\underline{u}\, |\eta^\sigma(\xi, \underline{u})|^2$ is positive and independent of ξ and σ, the limit $\mathfrak{A} \downarrow 0$ of the formal power series solution is the field corresponding to a Schwarzschild spacetime, whose future horizon is a level set of u, with $u < u_0$, when $|u_0| > 0$ is sufficiently small.

We use the notation

$$\Psi^\sigma_{\text{Picture}} \qquad \text{and} \qquad [\,\Psi^\sigma_{\text{Picture}}\,] \;=\; \sum_{k=0}^{\infty} \frac{1}{u^k}\,\Psi^\sigma_{\text{Picture}}(k)(\xi,\underline{u})$$

for the field and for the formal power series, where Picture $=$ R, H, F. Let RH, HR, FH, HF, FR, RF be the "transition matrices" between the different pictures, given by

$$\text{RH} = -\text{HR} = \operatorname{diag}(3,3,4,4,4,\quad 2,4,3,3,3,2,2,4,\quad 2,\quad 3,4,5,4) \tag{9.4a}$$
$$\text{FH} = -\text{HF} = \operatorname{diag}(4,4,8,8,8,\quad 0,4,4,4,4,4,4,8,\ -4,\quad 0,4,8,8) \tag{9.4b}$$
$$\text{FR} = -\text{RF} = \operatorname{diag}(1,1,4,4,4,\ -2,0,1,1,1,2,2,4,\ -6,-3,0,3,4) \tag{9.4c}$$

Observe that FR $=$ FH $+$ HR. Then (use: the figure in Subsection 9.1, equations (9.1), $a = \mathfrak{A}$ and Definitions 3.2, 3.4, 3.5)

$$\Psi^\sigma_{\text{H}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\text{HR}}\,\Psi^\sigma_{\text{R}}(\xi,\underline{u},\mathfrak{A}^2 u) \tag{9.5a}$$
$$\Psi^\sigma_{\text{F}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\text{FH}}\,\Psi^\sigma_{\text{H}}(\xi,\mathfrak{A}^{-4}\underline{u},\mathfrak{A}^{-4} u) \tag{9.5b}$$
$$\Psi^\sigma_{\text{F}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\text{FR}}\,\Psi^\sigma_{\text{R}}(\xi,\mathfrak{A}^{-4}\underline{u},\mathfrak{A}^{-2} u) \tag{9.5c}$$

The coefficient functions of the formal power series transform according to

$$\Psi^\sigma_{\text{H}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\text{HR}-2k}\,\Psi^\sigma_{\text{R}}(k)(\xi,\underline{u}) \tag{9.6a}$$
$$\Psi^\sigma_{\text{F}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\text{FH}+4k}\,\Psi^\sigma_{\text{H}}(k)(\xi,\mathfrak{A}^{-4}\underline{u}) \tag{9.6b}$$
$$\Psi^\sigma_{\text{F}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\text{FR}+2k}\,\Psi^\sigma_{\text{R}}(k)(\xi,\mathfrak{A}^{-4}\underline{u}) \tag{9.6c}$$

For all $k \geq 0$, we have:

$$\Psi^\sigma_{\text{R}}(k)(\xi,\underline{u}) \text{ is a polynomial in } \mathfrak{A} \tag{9.7a}$$
$$\Psi^\sigma_{\text{H}}(k)(\xi,\underline{u}) \text{ is a polynomial in } \mathfrak{A}^{-2} \text{ without constant term} \tag{9.7b}$$

Statement (9.7) is verified by induction over k. Just follow the construction of $[\,\Psi^\sigma_{\text{R}}\,]$ and $[\,\Psi^\sigma_{\text{H}}\,]$ in the proof of Lemma 6.1, keeping in mind that the coefficient functions, in the Regularized Picture, of the Minkowski background $[\,\mathcal{M}_{\mathfrak{A},\mathfrak{A}}\,]$ depend polynomially on $\mathfrak{A}$, and that $\textbf{DATA}^\sigma_R$ is independent of $\mathfrak{A}$. On the other hand, in the High Amplitude Picture, the Minkowski background $[\,\mathcal{M}_{1,1}\,]$ is independent of $\mathfrak{A}$, while $\textbf{DATA}^\sigma_H$ is proportional to $\mathfrak{A}^{-2}$. Incidentally, (9.7a) has already been shown in Remark 6.3. There, it was also shown that

$$\mathfrak{P}\text{-even } (\mathfrak{P}\text{-odd}) \text{ components of } \Psi^\sigma_{\text{R}}(k)(\xi,\underline{u}) \text{ are even (odd) polynomials in } \mathfrak{A} \tag{9.8}$$

The statement (9.8) also follows from (9.7a), (9.7b) and (9.6a), because the i-th component of $\Psi^\sigma_R(k)(\xi,\underline{u})$ is $\mathfrak{P}$-even ($\mathfrak{P}$-odd) if $(\text{RH})_{ii}$ is even (odd). Furthermore,

$$\text{the } i\text{-th component of } \Psi^\sigma_R(k)(\xi,\underline{u}) \text{ has degree} \leq (\text{RH})_{ii} - 2 + 2k \tag{9.9}$$

as a polynomial in $\mathfrak{A}$.

Lemma 9.1. *The 18 components of each $\Psi_F^\sigma(k)(\xi,\mathfrak{A}^4\underline{u})$, $k\geq 0$, are Laurent polynomials in $\mathfrak{A}^2$. If a component does not appear on the list*

$$\begin{aligned}
&(\omega_1)_F^\sigma(k)(\xi,\mathfrak{A}^4\underline{u}) && k=0,1\\
&(\omega_2)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u}) && \\
&(z_1)_F^\sigma(k)(\xi,\mathfrak{A}^4\underline{u}) && k=0,1,2,3\\
&(z_2)_F^\sigma(k)(\xi,\mathfrak{A}^4\underline{u}) && k=0,1\\
&(z_3)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u}) &&
\end{aligned}\tag{9.10}$$

then it is an actual polynomial in $\mathfrak{A}^2$ without constant term.

Proof. By (9.4b), (9.6b), (9.7b), all the components are Laurent polynomials in $\mathfrak{A}^2$. By (9.4c), (9.6c), (9.7a), all those not in (9.10) are polynomials without constant term. □

Lemma 9.2. *Suppose* (9.3)*. Define $U=\mathbb{R}^2\times(\frac{3}{4},\infty)$ and $\Lambda^\sigma(\xi)=\int_{1/2}^{3/4}\mathrm{d}\underline{u}|\eta^\sigma(\xi,\underline{u})|^2$. For $(\xi,\underline{u})\in U$, the functions in* (9.10) *are polynomials in $\mathfrak{A}^2$, and:*

$$\begin{aligned}
&(\omega_1)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u})=0 && (z_1)_F^\sigma(2)(\xi,\mathfrak{A}^4\underline{u})=2\,(\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}}-\boldsymbol{\lambda}_{1,1})\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma\\
&(\omega_1)_F^\sigma(1)(\xi,\mathfrak{A}^4\underline{u})=0 && (z_1)_F^\sigma(3)(\xi,\mathfrak{A}^4\underline{u})=\mathcal{O}(\mathfrak{A}^2)\\
&(\omega_2)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u})=-\Lambda^\sigma && (z_2)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u})=0\\
&(z_1)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u})=0 && (z_2)_F^\sigma(1)(\xi,\mathfrak{A}^4\underline{u})=-2\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}}\Lambda^\sigma\\
&(z_1)_F^\sigma(1)(\xi,\mathfrak{A}^4\underline{u})=0 && (z_3)_F^\sigma(0)(\xi,\mathfrak{A}^4\underline{u})=\Lambda^\sigma
\end{aligned}\tag{9.11}$$

where $2\frac{\partial}{\partial\overline{\xi}}=\frac{\partial}{\partial\xi^1}+i\frac{\partial}{\partial\xi^2}$ and $\mathbf{e}_{1,1}=\frac{1}{2}(1+|\xi|^2)$ and $\boldsymbol{\lambda}_{1,1}=-\frac{1}{2}(\xi^1+i\xi^2)$.

Proof. By (9.6c), the equations (9.11) can be translated from the Finite Mass to the Regularized Picture. In this proof, we work exclusively in the Regularized Picture. For convenience, we suppress the R and σ indices as well as the argument $(\xi,\underline{u})$. For example, $\omega_1(k)$ means $(\omega_1)_{\mathrm{R}}^\sigma(k)(\xi,\underline{u})$. We use the shorthands $\mathbf{e}=\mathbf{e}_{\mathfrak{A},\mathfrak{A}}$ and $\boldsymbol{\lambda}=\boldsymbol{\lambda}_{\mathfrak{A},\mathfrak{A}}$ (see, (4.3)). Equivalently, $\mathbf{e}=\mathfrak{A}\mathbf{e}_{1,1}$ and $\boldsymbol{\lambda}=\mathfrak{A}\boldsymbol{\lambda}_{1,1}$. For all the equations in (9.11) concerning *zeroth* order coefficient functions, use Definition 5.1 and (9.6c). We only note that on $\mathbb{R}^2\times(0,\infty)$,

$$z_3(0)\;=\;-4(\mathbf{e}\,\tfrac{\partial}{\partial\xi}+\overline{\boldsymbol{\lambda}})(\mathbf{e}\,\tfrac{\partial}{\partial\xi}+2\overline{\boldsymbol{\lambda}})\,\partial_{\underline{u}}^{-1}\eta-\eta\,\partial_{\underline{u}}^{-1}\overline{\eta}+\partial_{\underline{u}}^{-1}|\eta|^2$$

which reduces to $z_3(0)=\Lambda^\sigma$ on U. For the rest of (9.11), we use the equations

$$\begin{aligned}
N(z_1)+\tfrac{1}{u}D(z_2)&=\tfrac{1}{u^2}\big(Sz_1-2\boldsymbol{\lambda}z_2-\omega_6z_1\big)\\
&\qquad+\tfrac{1}{u^3}\big(2S\boldsymbol{\lambda}z_2-3\omega_1z_3+6\omega_3z_2+4\omega_8z_1-4\overline{\omega}_4z_2\big)\\
N(z_2)+\tfrac{1}{u}D(z_3)&=\tfrac{1}{u^2}\big(+2Sz_2-2\omega_6z_2\big)\\
&\qquad+\tfrac{1}{u^3}\big(-2\omega_1z_4+3\omega_3z_3+2\omega_8z_2-3\overline{\omega}_4z_3\big)\\
L(\omega_1)&=-z_1+\tfrac{1}{u}\,2\,\mathfrak{A}^2\,\omega_1+\tfrac{1}{u^2}\big(-2\,S\,\mathfrak{A}^2\,\omega_1-2\omega_1\omega_2\big)
\end{aligned}$$

The first and third appear (5.7a). For the second, we use (5.7a) and the constraint equation $y_1=0$. For the vector fields D, N, L, see (5.9). We obtain, in succession,

	on $\mathbb{R}^2 \times (0,\infty)$	on U
$z_1(1)$	$= -\mathfrak{A}^2 \underline{u}\partial_{\underline{u}}\eta - 2(\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})z_2(0)$	$= 0$
$z_2(1)$	$= 2\,\mathfrak{A}^2\,\underline{u}\,z_2(0) - 2\,\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}}\,z_3(0)$	$= -2\,\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma$
$\omega_1(1)$	$= -4(\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})(\mathbf{e}\,\frac{\partial}{\partial\xi} + 2\overline{\boldsymbol{\lambda}})\partial_{\underline{u}}^{-1}\eta$ $+ \mathfrak{A}^2\,\underline{u}\,\eta + \mathfrak{A}^2\,\partial_{\underline{u}}^{-1}\eta$	$= 0$
$z_1(2)$	$=$ (not needed)	$= 2\,(\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})\,\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma$
$z_1(3)$	$=$ (not needed)	$= \mathfrak{A}^2\underline{u}\,z_1(2) - \frac{2}{3}(\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})z_2(2)$

For $z_1(3)$, we have also used that $\omega_3(0)$, $\omega_4(0)$, $f_1(0)$, $f_2(0)$ all vanish on U (see, Definition 5.1). We know from (9.8) that $z_1(3)$ is a polynomial in $\mathfrak{A}^2$. It has no constant term on U, because $\mathbf{e}$, $\boldsymbol{\lambda}$ and $z_2(2)$ are odd polynomials in $\mathfrak{A}$. Now, use (9.6c). $\square$

Lemma 9.3. *Suppose* (9.3)*. Each component of each* $\Psi_F^\sigma(k)(\xi,\underline{u})$*,* $k \geq 0$*, is a Laurent polynomial in* $\mathfrak{A}^2$ *and a polynomial in* $\underline{u}$ *when* $\underline{u} > \frac{3}{4}\mathfrak{A}^4$*.*

Proof. By (9.3a) and the construction of the formal power series in the proof of Lemma 6.1, each $\Psi_{\mathrm{R}}^\sigma(k)(\xi,\underline{u})$ is a polynomial in $\underline{u}$ on $U = \mathbb{R}^2 \times (\frac{3}{4},\infty)$. Also recall (9.7a) and (9.8). The lemma now follows from (9.6c). $\square$

Let $\widehat{\Psi}_{\mathrm{F}}^\sigma(k)(\xi,\underline{u})$ be the polynomial extension in $\underline{u}$ of $\Psi_{\mathrm{F}}^\sigma(k)(\xi,\underline{u})$ from $\underline{u} > \frac{3}{4}\mathfrak{A}^4$ to $\underline{u} > 0$. Then, $[\,\widehat{\Psi}_{\mathrm{F}}^\sigma\,]$ is a Pole-Flip compatible pair of formal solutions to (5.7a) on $\mathbf{Strip}_\infty$ with Minkowski background $[\,\mathcal{M}_{1,1}\,]$, and $[\,\widehat{\Psi}_{\mathrm{F}}^{\sigma\,\sharp}\,] = 0$.

Lemma 9.4. *Suppose* (9.3)*. Then,* $\widehat{\Psi}_F^\sigma(0)(\xi,\underline{u})$ *is a polynomial in* $\mathfrak{A}^2$*. More precisely,*

$$\widehat{\Psi}_F^\sigma(0)(\xi,\underline{u}) = \tag{9.12}$$
$$\big(0,0,-\underline{u}\Lambda^\sigma,0,0,\;\; 0,-\Lambda^\sigma,0,0,0,0,0,\underline{u}\Lambda^\sigma,\;\; 0,0,\Lambda^\sigma,2\,\underline{u}\,\mathbf{e}_{1,1}\,\tfrac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma,0\big) + \mathcal{O}(\mathfrak{A}^2)$$

Proof. By direct calculation. $\square$

Proposition 9.3. *Suppose* (9.3)*. Each component of each* $\widehat{\Psi}_F^\sigma(k)(\xi,\underline{u})$*,* $k \geq 0$*, is simultaneously a polynomial in* $\underline{u}$ *and* $\mathfrak{A}^2$*, for all* $(\xi,\underline{u}) \in \mathbb{R}^2 \times (0,\infty)$*.*

Proof. They are polynomials in $\underline{u}$ by definition. The case $k = 0$ is covered by Lemma 9.4. The general case is shown by induction over k, using the fact that the equations (6.2) hold with Minkowski background $[\,\mathcal{M}_{1,1}\,]$. In the present case, (6.2) are equations for polynomials in $\underline{u}$. The generic terms $\mathcal{P}_k$ on the right hand sides in (6.2) are, by the inductive hypothesis, polynomials in $\mathfrak{A}^2$. When using (6.2a) through (6.2r) in this order to determine the components of $\widehat{\Psi}_F^\sigma(k)$, only polynomials in $\mathfrak{A}^2$ are generated. If $\frac{\partial}{\partial\underline{u}}$ appears on the left hand side, then the non-constant terms as a $\underline{u}$-polynomial of the corresponding component of $\widehat{\Psi}_F^\sigma(k)$ are determined uniquely by the right hand side. The constant term of integration is determined by the restriction of $\widehat{\Psi}_F^\sigma$ to $\underline{u} = \mathfrak{A}^4$, that is $\widehat{\Psi}_F^\sigma(k)(\xi,\mathfrak{A}^4)$, which is itself a polynomial in $\mathfrak{A}^2$, by Lemmas 9.1 and 9.2. $\square$

Proposition 9.4. *Suppose* (9.3)*. For each* $k \geq 0$*, let* $\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma(k)$ *be the constant term of* $\widehat{\Psi}_F^\sigma(k)$ *as a polynomial in* $\mathfrak{A}^2$*. Then* $[\,\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma\,]$ *is the unique formal solution to* (5.7a)

with Minkowski background $[\mathcal{M}_{1,1}]$ *and characteristic initial data*

$$\widehat{\Psi}^{\sigma}_{F,\mathfrak{A}=0}(0)(\xi,\underline{u}) =$$
$$\big(0,0,-\underline{u}\Lambda^{\sigma},0,0,\;\; 0,-\Lambda^{\sigma},0,0,0,0,0,\underline{u}\Lambda^{\sigma},\;\; 0,0,\Lambda^{\sigma},2\,\underline{u}\,\mathbf{e}_{1,1}\,\tfrac{\partial}{\partial\xi}\,\Lambda^{\sigma},0\big) \tag{9.13a}$$

$$[\,\widehat{\Psi}^{\sigma}_{F,\mathfrak{A}=0}\,](\xi,0,u) = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \oplus \begin{pmatrix}0\\-\Lambda^{\sigma}\\0\\ \vdots\\0\end{pmatrix} \oplus \begin{pmatrix}\frac{2}{u^2}(\mathbf{e}_{1,1}\,\frac{\partial}{\partial\xi} - \boldsymbol{\lambda}_{1,1})\mathbf{e}_{1,1}\,\frac{\partial}{\partial\xi}\,\Lambda^{\sigma}\\ -\frac{2}{u}\mathbf{e}_{1,1}\,\frac{\partial}{\partial\xi}\,\Lambda^{\sigma}\\ \Lambda^{\sigma}\\ 0\\ 0\end{pmatrix} \tag{9.13b}$$

Its coefficient functions are polynomials in $\underline{u}$*. Moreover,* $[\,\widehat{\Psi}^{\sigma\,\sharp}_{F,\mathfrak{A}=0}\,] = 0$*.*
Particularly, if $\Lambda^{\sigma}(\xi) \equiv \Lambda$ *is independent of* ξ *and* σ*, then it represents Schwarzschild spacetime, with mass* $m = 2^{-3/2}\Lambda$*.*

Proof. The first part follows from Lemma 9.1, 9.2, 9.3, 9.4 and Proposition 9.3. In particular, (9.13a) follows from (9.12), and (9.13b) follows from (9.11).
If $\Lambda^{\sigma}(\xi) \equiv \Lambda$ is independent of ξ and σ, then $\widehat{\Psi}^{\sigma}_{\mathrm{F},\mathfrak{A}=0}(0)(\xi,\underline{u})$ and $[\,\widehat{\Psi}^{\sigma}_{\mathrm{F},\mathfrak{A}=0}\,](\xi,0,u)$ in (9.13a) and (9.13b) correspond to spherically symmetric initial data for (5.7a) with Minkowski background $[\,\mathcal{M}_{1,1}\,]$. Therefore, $[\,\widehat{\Psi}^{\sigma}_{\mathrm{F},\mathfrak{A}=0}\,]$ is spherically symmetric, and therefore, by a formal Birkhoff Theorem, a formal expansion of a Schwarzschild vacuum field. The component γ_2 of $[\,\mathcal{M}_{1,1}\,] + u^{-M}\,[\,\widehat{\Psi}^{\sigma}_{\mathrm{F},\mathfrak{A}=0}\,]$ vanishes on the sphere $(u,\underline{u}) = (-\Lambda,0)$, and therefore this sphere is a section of the Schwarzschild horizon. Its area is equal to $2\pi\Lambda^2$, which gives the formula for the mass m. □

The discussion of the formal solution $[\,\Psi^{\sigma}_{\mathrm{F}}\,]$ is finished. We now turn to classical solutions. The bound (8.34) implies that for all $u < -\mathbf{c}^{-1}$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^{\alpha}\Big(\Psi^{\sigma}_{\mathrm{R}}(\,\cdot\,,u) - \sum_{k=0}^{K}\frac{\widehat{\Psi}^{\sigma}_{\mathrm{R}}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4},1)\right)} \;\le\; \frac{1}{\mathbf{c}\,|u|^{K+1}}$$

This bound, in turn, implies the Finite Mass Picture bound (use (9.5c), (9.6c))

$$\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^{\alpha}\Big(\Psi^{\sigma}_{\mathrm{F}}(\,\cdot\,,u) - \sum_{k=0}^{K}\frac{\widehat{\Psi}^{\sigma}_{\mathrm{F}}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)\right)} \;\le\; \frac{\mathfrak{A}^{2K-4R+8}}{\mathbf{c}\,|u|^{K+1}}$$

when $u < -\mathfrak{A}^2\mathbf{c}^{-1}$. The power of $\mathfrak{A}$ on the right hand side arises as $2K-4R+8 = 2(K+1)-4(R-3)-6$. Given $R \ge 4$, we choose $K = 2R-3$. (Then, the constant $\mathbf{c}$ depends only on R and $\max_{\sigma\in\{-,+\}}\|\eta^{\sigma}\|_{C^{5R}(D_4(0)\times(0,2))}$.) Altogether, we obtain:

Proposition 9.5. *Suppose* (9.3)*. For each* $u_0 < 0$ *and each* $R \ge 4$*, the limit as* $\mathfrak{A}\downarrow 0$ *of*

$$\sup_{u<u_0}\;|u|^{2R-2}\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^{\alpha}\Big(\Psi^{\sigma}_{F}(\,\cdot\,,u) - \sum_{k=0}^{2R-3}\frac{\widehat{\Psi}^{\sigma}_{F}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)\right)}$$

is zero. Here, the solution Ψ^{σ}_{F}*, the functions* $\widehat{\Psi}^{\sigma}_{F}(k)$*, and the* $\underline{u}$*-interval* $(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)$ *depend on* $\mathfrak{A}$*. Under appropriate conditions (see, Proposition 9.4), the Schwarzschild vacuum field can be approximated arbitrarily closely on the strip* $\mathbf{III}_1$*.*

Remark 9.7. To obtain the last result, we had to explicitly calculate the first four orders of the far field expansion, in particular for the component z_1.

So far, we have provided complete, detailed arguments for each of our statements. At this point of the paper, the character of our discussion changes. For the rest of Section 9, we sketch additional applications of our overall hybrid method and give informal arguments to support our informal assertions. We will give rigorous discussions in another place.

9.5. 2D Limit in the Regularized Picture: beyond the far field regime. Theorem 8.1 produces vacuum fields on $\mathbf{Strip}(1, \mathbf{c})$, with $\mathbf{c} > 0$ independent of $\mathfrak{A}$ (see, Remark 9.1). In this subsection, we informally answer the question: How can one control these vacuum fields when $u > -\frac{1}{\mathbf{c}}$ and get closer to the (expected) singularity? The strategy, as before, is to generate an appropriate formal solution to the initial value problem (see, Section 5), and then to construct a classical solution by estimating its deviation from a truncation of the formal solution. The far field expansion has run its course. We need an expansion in $\mathfrak{A}^2$.

In this subsection, we make the 2D Limit Assumptions (see, Definition 9.1), work exclusively in the Regularized Picture, and suppress the R and σ indices. For example, Φ means Φ_R^σ.

In analogy with (5.5a), (5.5c), (5.6a), (5.6b), set

$$
\begin{aligned}
P &= \operatorname{diag}(1,1,0,2,2) \oplus \operatorname{diag}(0,0,1,1,1,0,0,0) \oplus \operatorname{diag}(0,1,0,1,0) && (9.14a)\\
P^\sharp &= \operatorname{diag}(2,2,1,1,1) \oplus \operatorname{diag}(1,0,0,0,0,0,1,1,1) \oplus \operatorname{diag}(1,0,1) && (9.14b)\\
\Phi &= \mathfrak{A}^P \left(\mathfrak{A}^{-P}\Phi\right) && (9.14c)\\
\Phi^\sharp &= \mathfrak{A}^{P^\sharp} \left(\mathfrak{A}^{-P^\sharp}\Phi^\sharp\right) && (9.14d)
\end{aligned}
$$

The third line indicates that we wish to write Φ as $\mathfrak{A}^P$ times a *new field*. In order not to introduce yet another name, we write the *new field* as $\mathfrak{A}^{-P}\Phi$. Similar for $\mathfrak{A}^{-P^\sharp}\Phi^\sharp$. We make formal expansions of $\mathfrak{A}^{-P}\Phi$ in powers of $\mathfrak{A}^2$. The properties of the far field expansion, for instance (9.7a) and (9.8), suggest that the ansatz (9.14) is consistent. However, this must be checked.

A formal power series $\{\, f \,\}$ in $\mathfrak{A}^2$ on an open subset $\mathcal{U} \subset \mathbf{Strip}_\infty$ with values in a vector space X is a formal sum

$$\{\, f \,\} \;=\; \sum_{\ell=0}^{\infty} \left(\mathfrak{A}^2\right)^\ell f\{\ell\}(x) \tag{9.15}$$

For each $\ell \geq 0$, the coefficient $f\{\ell\} : \mathcal{U} \to X$ is smooth and independent of $\mathfrak{A}$.

Let $\{\, \mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}} \,\}$ be the formal expansion of $\mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ in powers of $\mathfrak{A}^2$ (see, Definition 4.1). It is defined on $\mathbf{Strip}_\infty$ and takes values in $\mathcal{R}$.

Our ansatz is to write the field $\mathfrak{A}^{-P}\Phi$ as a formal series $\{\, \mathfrak{A}^{-P}\Phi \,\}$ on some open set $\mathcal{U} \subset \mathbf{Strip}_\infty$ with values in $\mathcal{R}$. In this context, the far field ansatz (Section 5) becomes

$$\left\{\, \mathfrak{A}^{-P}\Phi \,\right\} \;=\; \left\{\, \mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}} \,\right\} + u^{-M} \left\{\, \mathfrak{A}^{-P}\Psi \,\right\} \tag{9.16}$$

To define the associated formal constraint field, see Definition 2.4 and (5.6b), we fix the weight functions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by (5.6c), as before. Then, by direct inspection,

$\{\, \mathfrak{A}^{-P^\sharp} \Phi^\sharp \,\}$ or, equivalently, $\{\, \mathfrak{A}^{-P^\sharp} \Psi^\sharp \,\}$ are also formal power series of the form (9.15). They are defined on $\mathcal{U}$ and take values in $\widehat{\mathcal{R}}$. For each $\ell \geq 0$, the coefficient $(\mathfrak{A}^{-P^\sharp} \Phi^\sharp)\{\ell\}$ is determined by $(\mathfrak{A}^{-P} \Phi)\{m\}$, $0 \leq m \leq \ell$. Similar for $(\mathfrak{A}^{-P^\sharp} \Psi^\sharp)\{\ell\}$.

We want to formally solve the same *characteristic initial value problem* as before:

- (5.7a) with Ψ and $\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ replaced by $\mathfrak{A}^P \{\, \mathfrak{A}^{-P} \Psi \,\}$ and $\mathfrak{A}^P \{\, \mathfrak{A}^{-P} \mathcal{M}_{\mathfrak{A},\mathfrak{A}} \,\}$,
- $\{\, \mathfrak{A}^{-P^\sharp} \Psi^\sharp \,\} = 0$,
- formal asymptotic initial conditions (5.3a), (5.3b) with, say, $\underline{u}_0 = \frac{1}{2}$.

It is understood that **DATA** $= \eta$ (see, Definition 5.1) is fixed and independent of $\mathfrak{A}$.

Remark 9.8. The meaning of the formal asymptotic initial condition (5.3a) is

$$\sum_{\ell=0}^{\infty} \big(\mathfrak{A}^2\big)^{\ell} \lim_{u \to -\infty} \big(\mathfrak{A}^{-P} \Psi\big)\{\ell\}(\xi, \underline{u}, u) \;=\; \mathfrak{A}^{-P}\, \Psi(0)(\xi, \underline{u}).$$

Equality is in the sense of $\mathcal{R}$-valued formal power series in $\mathfrak{A}^2$. The right hand side is actually a polynomial in $\mathfrak{A}^2$ of degree 1. The function $\Psi(0)$ is given in Definition 5.1.

Constructing a solution $\{\, \mathfrak{A}^{-P} \Psi \,\}$ to this formal initial value problem requires solving an infinite family of differential equations. All but a finite number of them are linear. It is possible to arrange these equations so that, when they are solved step by step, the "angular derivatives" $\frac{\partial}{\partial \xi^1}$, $\frac{\partial}{\partial \xi^2}$ are only applied to functions that have already been constructed. An essential ingredient is

$$D = \mathcal{O}(\mathfrak{A}) \qquad N = \tfrac{\partial}{\partial u} + \mathcal{O}(\mathfrak{A}^2) \qquad L = (1 + \tfrac{1}{u^2} f_3)\, \tfrac{\partial}{\partial \underline{u}} \tag{9.17}$$

as $\mathfrak{A} \to 0$. Here, $\mathcal{O}(\mathfrak{A}^k)$ stands for $\mathcal{O}(\mathfrak{A}^k) \frac{\partial}{\partial \xi^1} + \mathcal{O}(\mathfrak{A}^k) \frac{\partial}{\partial \xi^2}$, when $k = 1, 2$. In this sense, one only has to solve 2-dimensional problems in the $(\underline{u}, u)$ plane.

Observe that property (9.9) of the $\frac{1}{u}$ expansion implies that the coefficient function $(\mathfrak{A}^{-P} \Psi)\{\ell\}$, $\ell \geq 1$ is of the order $\mathcal{O}(|u|^{-\ell+1})$ as $u \to -\infty$. For this reason, we expect that all the arguments in Section 8 can be applied, with minor modifications, when the function Ψ_K in **(S2)** is replaced by the truncation

$$\mathfrak{A}^P \sum_{\ell=0}^{K+2} \big(\mathfrak{A}^2\big)^{\ell} \big(\mathfrak{A}^{-P} \Psi\big)\{\ell\}$$

of $\mathfrak{A}^P \{\, \mathfrak{A}^{-P} \Psi \,\}$. One should be able to conclude, in analogy with Theorem 8.1, that both a classical solution Ψ and a formal power series solution $\mathfrak{A}^P \{\, \mathfrak{A}^{-P} \Psi \,\}$ exist on **Strip**$(1, \mathbf{c})$, and that

$$\mathfrak{A}^{-P} \Psi \;\; - \;\; \sum_{\ell=0}^{K+1} \big(\mathfrak{A}^2\big)^{\ell} \big(\mathfrak{A}^{-P} \Psi\big)\{\ell\}, \tag{9.18}$$

and all its partial derivatives up to some finite order, are estimated, in absolute value, by $\leq \mathbf{c}^{-1}\, \mathfrak{A}^{2K+4}\, |u|^{-K-1}$ on **Strip**$(1, \mathbf{c})$. Here, smallness conditions similar to those in Theorem 8.1 must be made. In particular, $\mathbf{c} > 0$ has to be sufficiently small.

The fact that the difference (9.18) goes to zero as $\mathfrak{A} \downarrow 0$, uniformly on **Strip**$(1, \mathbf{c})$, means that the formal $\mathfrak{A}^2$ expansion "has not yet been exhausted". To better understand

what happens, let us examine the formal expansion in just a little more detail. We only discuss the zeroth coefficient, $(\mathfrak{A}^{-P}\Psi)\{0\}$ or, equivalently, $(\mathfrak{A}^{-P}\Phi)\{0\}$, see (9.16). Set $a = \gamma_2\{0\}/e_3\{0\}$ and $b = \gamma_6\{0\}$. The constraint equations $u_2\{0\} = u_3\{0\} = u_6\{0\} = 0$ and the equation $\frac{\partial}{\partial u} e_3\{0\} = 2e_3\{0\}\,\Re\gamma_8\{0\}$ derived from (5.7a) yield the system $\frac{\partial}{\partial u} a = -2ab$ and $\frac{\partial}{\partial \underline{u}} b = -2ab$. The initial conditions (see, Remark 9.8)

	$e_3\{0\}$	$\omega_2\{0\} = u^2\,\gamma_2\{0\}$	$\omega_6\{0\} = u^2(\gamma_6\{0\} - \frac{1}{u})$
$\underline{u} < \frac{1}{2}$	1	0	0
$u \to -\infty$	1	$-\partial_{\underline{u}}^{-1}\vert\mathbf{DATA}\vert^2$	0

select the unique solution

$$\frac{\gamma_2\{0\}}{e_3\{0\}} = \frac{1}{2h}\,\frac{\partial}{\partial \underline{u}} h \qquad \gamma_6\{0\} = \frac{1}{2h}\,\frac{\partial}{\partial u} h \qquad h(\xi, \underline{u}, u) = u^2 - \big(\varphi(\xi, \underline{u})\big)^2 \tag{9.19}$$

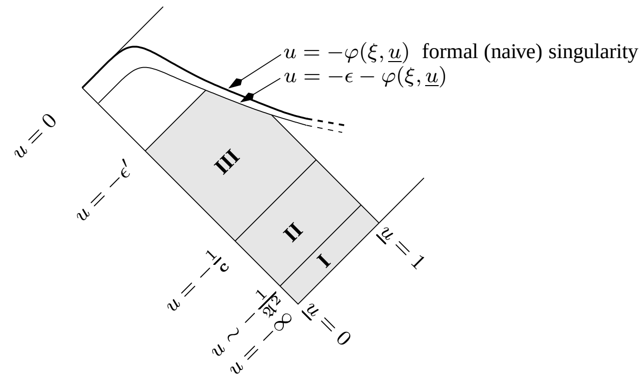
where $\varphi \geq 0$ and $\varphi^2 = 2\,\partial_{\underline{u}}^{-1}\partial_{\underline{u}}^{-1}|\mathbf{DATA}|^2$. The solution (9.19) is defined on $\mathcal{U}_0$, where

$$\mathcal{U}_\epsilon \;=\; \big\{(\xi, \underline{u}, u) \in \mathbf{Strip}_\infty \;\big|\; u < -\epsilon - \varphi(\xi, \underline{u})\big\}$$

for every $\epsilon \geq 0$. The solution (9.19) and therefore the formal solution $\{\,\mathfrak{A}^{-P}\Phi\,\}$ break down at $u = -\varphi(\xi, \underline{u}) < 0$, for example because $\gamma_6\{0\}$ diverges. Conversely, it can be shown that the whole formal solution $\{\,\mathfrak{A}^{-P}\Phi\,\}$ to the initial value problem exists on $\mathcal{U}_0$, that is, *no "earlier" breakdown occurs*. At the formal level, the scalar curvature invariants are in general unbounded as $u \uparrow -\varphi(\xi, \underline{u}) < 0$. We refer to $u = -\varphi(\xi, \underline{u}) < 0$ as the *formal (naive) singularity*. Observe that $\gamma_2\{0\} \leq 0$ and $\gamma_6\{0\} < 0$ on $\mathcal{U}_0$.

It follows from the structure of the matrix P, in particular its nonzero entries, that the components $e_3\{0\}$, $\gamma_1\{0\}$, $\gamma_2\{0\}$, $\gamma_6\{0\}$, $\gamma_7\{0\}$, $\gamma_8\{0\}$, $w_1\{0\}$, $w_3\{0\}$, $w_5\{0\}$ of the coefficient function $(\mathfrak{A}^{-P}\Phi)\{0\}$ satisfy the quasilinear symmetric hyperbolic system and the constraints in Proposition 2.4. This system has been investigated in situations with higher symmetry, for example in [Sze]. In our present context, however, the fields depend on all four coordinates. The collapse of the frame as $\mathfrak{A} \to 0$, see (9.17), is responsible for reducing the four-dimensional system to a family of two-dimensional systems, one for each ξ. It is possible to quasi-explicitly solve these two-dimensional systems near the formal singularity. That is, there is a formal solution given by an appropriate expansion in the "distance" from the formal singularity which is an asymptotic expansion to the true classical solution (of the two-dimensional system). The behavior of the solution to this two-dimensional system leads us to speculate that the $\mathfrak{A}^2$ expansion exhibits an instability close to the formal singularity. This instability appears to drive the full four dimensional system into a new regime in which the classical vacuum solution may display features of the BKL scenario. See, [BKL] and references therein.

We conclude this subsection with a discussion of the figure



- Christodoulou [Chr] constructs strongly focused gravitational wave solutions on **I** (see, Remark 9.3). Recall that $u \sim -\frac{1}{\mathfrak{A}^2}$ is the place where trapped spheres first form, see Proposition 9.2.
- In this paper, the far field expansion has been used to construct vacuum fields on the larger $\mathbf{I} \cup \mathbf{II} = \mathbf{Strip}(1, \mathbf{c})$, where $\mathbf{c} > 0$ is sufficiently small. See, Theorem 8.1.
- The $\mathfrak{A}^2$ expansion outlined in this subsection allows one, using appropriate energy estimates, to construct classical vacuum fields on at least $\mathbf{I} \cup \mathbf{II} \cup \mathbf{III} = \mathbf{Strip}(1, \epsilon'^{-1}) \cap \mathcal{U}_\epsilon$. Here, $\epsilon, \epsilon' > 0$ are arbitrary constants (in the figure, $0 < \epsilon < \epsilon'$), and $|\mathfrak{A}|$ is sufficiently small, depending on ϵ, ϵ'. Moreover, the formal $\mathfrak{A}^2$ power series solution is an asymptotic expansion to the true classical solution as $\mathfrak{A} \to 0$, uniformly on $\mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$. In other words, the $\mathfrak{A}^2$ expansion allows one to construct and control the solution up to any "finite distance" from the formal singularity.
  The justification of the last statement relies on the fact that a suitable truncation of the $\mathfrak{A}^2$ expansion is an approximate vacuum field, with error terms going to zero uniformly on $\mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$ as $\mathfrak{A} \to 0$ (by a compactness argument). Furthermore, these error terms decay quickly enough as $u \to -\infty$ to be "integrable". See, the discussion of (9.18).

Remark 9.9. Motivated by [BK] and [AnRe], we expect that coupling the gravitational field to a massless scalar field will make it possible to construct, under suitable generic conditions, strongly focused solutions from past null infinity all the way into a piece of the singularity.

Remark 9.10. Observe that:

- One only has to solve *linear* equations to inductively construct the far field expansion $[\,\Psi\,]$ in Lemma 6.1.
- By contrast, the construction of the leading term of the $\mathfrak{A}^2$ expansion $\{\,\mathfrak{A}^{-P}\Psi\,\}$ requires the solution of a *nonlinear* (effectively two-dimensional) system. In other words, one is expanding around a non-trivial "background".

9.6. 4D Limit. Recall that in the 2D Limit, the $\mathfrak{P}$-odd components of the coefficients of the far field formal solution $[\,\Phi_R\,]$ go to zero as $\mathfrak{A} \to 0$ (in the Regularized Picture). On the other hand, the $\mathfrak{P}$-even components do not in general go to zero. See, Remark 6.3 or (9.8). This asymmetric feature of the 2D Limit disappears in the more general 4D Limit (see, Subsection 9.2), as one can see by looking at the far field expansion. It

is for the purpose of taking the 4D Limit that the *two* scaling parameters, a and $\mathfrak{A}$, have been carried along through the whole paper.

To compare the 4D Limit with the 2D Limit, it is useful to formulate the second smallness condition in (8.32) in a picture in which the Minkowski background and the stereographic coordinates ξ are fixed (independent of a and $\mathfrak{A}$), for example the High Amplitude Picture. The smallness condition becomes

$$\max_{\sigma\in\{-,+\}} \left\| \partial^\alpha \,\mathbf{DATA}^\sigma_H \right\|_{C^0(D_4(0)\times(0,1))} \;\le\; \frac{\mathbf{b}}{\mathfrak{A}^2}\left|\frac{a}{\mathfrak{A}}\right|^{\alpha_1+\alpha_2}.$$

for all $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| \le R+4$. Notice that in the 4D Limit, $|\frac{a}{\mathfrak{A}}|$ is a large factor, and there is one factor for each "angular derivative".

A. Index of Notation

This is a partial list of symbols used in this paper. *It refers to their main / typical usage. Warning: These symbols can have different meanings. However, these meanings will always be made clear in each particular context.* (For example, the entry for the symbol S in the index below refers to equation (4.4), which corresponds to the main / typical meaning of the symbol S in many sections of this paper. Nevertheless, the symbol S stands for a field transformation in Section 3, and it stands for a set in the local context of Proposition 7.1.) Symbols which only appear in the Appendices are not listed. In the third column, a selected reference is given.

Symbol	Typical Meaning	See
$(\star)$, $(\star\star)$	frame nondegeneracy conditions	Definition 2.1
$[A,B] = AB - BA$	commutator of operators	
$\lesssim_p$	parameter p dependent bound	Convention 7.1
$\mathfrak{A}$, a	scaling parameters	Definitions 3.5, 4.1
$\mathfrak{A}$, $\mathfrak{J}$, $\mathfrak{Z}$, $\mathfrak{C}$	field transformations	Section 3
$\mathbf{A}(\Phi)$, $\mathbf{f}(\Phi)$	constituents of **(SHS)**	Definition 2.3
$\mathbf{A}(\Psi,x)$, $\mathbf{f}(\Psi,x)$	constituents of (5.7a)	Proposition 5.1
$\widehat{\mathbf{A}}(\Phi)$, $\widehat{\mathbf{f}}(\Phi,\partial_x\Phi)$	constituents of $\widehat{\textbf{(SHS)}}$	Proposition 2.3
$\widehat{\mathbf{A}}(\Psi,x)$, $\widehat{\mathbf{f}}(\Psi,\partial_x\Psi,x)$	constituents of (5.7b)	Proposition 5.1
$\mathbf{B}\Xi = Q\Xi + \mathbf{Src}$		**(S4)** in Section 8
$\widehat{\mathbf{B}}\Xi^\sharp = \widehat{Q}\Xi^\sharp$		$\widehat{\textbf{(S2)}}$ in Section 8
$\mathbf{b}$, $\mathbf{c}$	small constants	Theorem 8.1
C	complex conjugation operator	Convention 8.3
$\mathfrak{d}$		Convention 7.3
DATA	data at past null infinity	Definition 5.1
$D, \overline{D}, N, L$	complex frame vector fields	Definition 2.1
D_r, B_r	open disk and open ball	Convention 7.1
$\mathbf{e}$, $\boldsymbol{\lambda}$, ρ		Definition 4.1
$E^k_{\mathcal{X}}\{f\}(t)$	energy	Definition 7.1
$\Phi = (e,\gamma,w)$	$\mathcal{R}$ valued field	Section 2
$\Phi^\sharp = (t,u,v)$	$\widehat{\mathcal{R}}$ valued constraint field	Definition 2.4
$\mathbf{Flip}_\alpha$	Pole-Flip transformation	Definition 3.6
$\mathcal{I}$, $\mathscr{J}$	intervals	
$\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\ldots$, $\mathcal{G}^\sharp$, $\ldots$	generic symbols	Definitions 8.1, 8.2

Symbol	Typical Meaning	See
j^μ	energy current vector field	Section 7
K	truncation index, see also 'Ψ_K'	Theorem 8.1
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	weight functions	Definition 2.3
M	far field ansatz matrix	Section 5
$\mathcal{M}_{a,\mathfrak{A}}$, $[\mathcal{M}_{a,\mathfrak{A}}]$	doubly scaled Minkowski field	Definitions 4.1, 5.2
$\mathbf{M}(q,\Theta)\Theta = h(q,\Theta)$	general symm. hyp. system	Section 7
$\mathbf{M}(q,\Xi)\Xi = h(q,\Xi)$	a particular symm. hyp. system	**(S10)** in Section 8
$\mathbb{N}$	set of integers > 0	
$\mathbb{N}_0$	set of integers ≥ 0	
$\mathcal{O}(b)$, $\mathcal{O}(\xi_0, b, t)$	families of subsets of $\mathbb{R}^3$	**(E0)**, **(RE0)** in Sec. 7
π, $\widehat{\pi}$	certain permutation matrices	**(S3)**, $\widehat{\textbf{(S1)}}$ in Sec. 8
$\mathfrak{P}$	parity field transformation	Remark 2.9
$\mathcal{P}$, $\mathcal{P}^\sharp$, $\mathcal{P}_k$	generic symbols	Definitions 5.3, 6.1
∂^α	multi-derivative	Definition 7.1
$\partial_{\underline{u}}^{-1}$	an integration operator	(5.2)
∂_x, ∂_q, $\partial_{\mathbf{q}}$	gradient operator w.r.t. x, q, $\mathbf{q}$	
$\frac{\partial}{\partial \xi}$	$= \frac{1}{2}(\frac{\partial}{\partial \xi^1} - i \frac{\partial}{\partial \xi^2})$	Definition 5.1
$q = (q^0, q^1, q^2, q^3)$ $= (t, \xi^1, \xi^2, \underline{u})$	coordinates	Convention 7.1
$\mathbf{q} = (q^1, q^2, q^3)$	spatial components of q	Convention 7.1
$\mathcal{Q}$, $\mathcal{K}$	general subsets of $\mathbb{R}^3$	**(EB4)** in Section 7
$\mathcal{Q}$, $\mathcal{K}$	particular subsets of $\mathbb{R}^3$	**(S9)** in Section 8
R, H, F	three pictures	Section 9
R	differentiability index	Theorem 8.1
$\mathcal{R}$, $\widehat{\mathcal{R}}$	real vector spaces	(2.1), (2.6)
$\Re$, $\Im$	real / imaginary part operators	
$\mathbb{R}$, $\mathbb{C}$	the real and complex numbers	
S		(4.4)
Sup$_{\mathcal{X}}^{(k)}\{f\}(t)$	supremum norm	Definition 7.1
(SHS) , $\widehat{\textbf{(SHS)}}$, **(subSHS)**	symmetric hyperbolic systems	Section 2
Strip(μ, λ)	family of open subsets of $\mathbb{R}^4$	(4.1)
σ	stereographic chart superscript	Proposition 6.4
t	time coordinate, equal to $u + \underline{u}$	Convention 7.1
$\mathcal{U}$	general open subset of $\mathbb{R}^4$	Section 2
Ξ, $\Xi^\sharp$	$\pi^{-1}\mathcal{R}$ and $\widehat{\pi}^{-1}\widehat{\mathcal{R}}$ valued fields	**(S3)**, $\widehat{\textbf{(S1)}}$ in Section 8
$x = (x^1, x^2, x^3, x^4)$	coordinates	Section 2
$= (\xi^1, \xi^2, \underline{u}, u)$	coordinates	Section 2
ξ	either (ξ^1, ξ^2) or $\xi^1 + i\xi^2$	
$\overline{z} = \Re z - i\, \Im z$	complex conjugation	
$\Psi = (f, \omega, z)$	$\mathcal{R}$ valued field	(5.1)
$[\Psi]$, $\Psi(k)$	formal power series, coefficients	(6.1)
Ψ_K	truncated formal power series	**(S2)** in Section 8
$\Psi^\sharp = (s, p, y)$	$\widehat{\mathcal{R}}$ valued constraint field	Proposition 5.1
$[\Psi^\sharp]$, $\Psi^\sharp(k)$	formal power series, coefficients	Section 6

B. Generalized Vacuum Equations

Our main reference for this appendix is [Fr].

The vacuum Einstein equations, written in local coordinates x^1, x^2, x^3, x^4 on a connected open set $\mathcal{U}$ in $\mathbb{R}^4$, are a nonlinear system of partial differential equations for the ten metric tensor fields $g_{\mu\nu}$. Namely,

$$R^{\gamma}{}_{\alpha\gamma\beta} \;=\; 0 \tag{B.1}$$

where,

$$R^{\delta}{}_{\alpha\gamma\beta} \;=\; \tfrac{\partial}{\partial x^{\gamma}}\varGamma^{\delta}_{\beta\alpha} - \tfrac{\partial}{\partial x^{\beta}}\varGamma^{\delta}_{\gamma\alpha} + \varGamma^{\mu}_{\beta\alpha}\varGamma^{\delta}_{\gamma\mu} - \varGamma^{\mu}_{\gamma\alpha}\varGamma^{\delta}_{\beta\mu} - \big(\varGamma^{\mu}_{\gamma\beta} - \varGamma^{\mu}_{\beta\gamma}\big)\varGamma^{\delta}_{\mu\alpha}$$

are the components of the Riemann curvature tensor for the Levi-Civita connection $\varGamma^{\gamma}_{\mu\nu} = g^{\gamma\lambda}\varGamma_{\mu\nu\lambda}$:

$$\varGamma_{\mu\nu\lambda} \;=\; \tfrac{1}{2}\big(\tfrac{\partial}{\partial x^{\mu}}g_{\nu\lambda} + \tfrac{\partial}{\partial x^{\nu}}g_{\mu\lambda} - \tfrac{\partial}{\partial x^{\lambda}}g_{\mu\nu}\big)$$

associated to the metric $g_{\mu\nu}$.

There are patent mathematical advantages to introducing more fields and equations that, in the presence of appropriate constraints, collapse to the vacuum Einstein equations. The purpose of this appendix is to introduce a particular generalized system of vacuum equations and explain how it will be used. In this appendix and in Appendix C, we work with real quantities. In Appendix D we employ a complex tetrad formalism, see [NP], as in the main body of this paper.

Definition B.1. *A* **generalized spacetime** *is an open subset $\mathcal{U}$ of $\mathbb{R}^4$ together with*

- *16 frame fields $E_a{}^{\mu}$ and the associated vector fields*

$$E_a \;=\; E_a{}^{\mu}\tfrac{\partial}{\partial x^{\mu}}$$

  It is assumed that E_1, E_2, E_3, E_4 are frame vector fields. That is, they are linearly independent tangent vectors at every point of $\mathcal{U}$.
- *A constant, symmetric matrix g_{ab} with three positive and one negative eigenvalues. For our purposes, the matrix g_{ab} and its inverse g^{ab} are*

$$\big(g_{ab}\big) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \big(g^{ab}\big) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{B.2}$$

- *24 connection fields $\varGamma_{abc}$ that are antisymmetric in the indices b and c.*
- *10 Weyl fields $W_{abk\ell}$ characterized by*

$$\begin{aligned} W_{abk\ell} &= -W_{bak\ell} & \qquad W_{ajk\ell} + W_{a\ell jk} + W_{ak\ell j} &= 0 \\ W_{abk\ell} &= -W_{ab\ell k} & g^{ak}W_{abk\ell} &= 0 \\ W_{abk\ell} &= W_{k\ell ab} & & \end{aligned}$$

Convention B.1. Small Latin indices a, b, c … are frame indices and always run from one to four. Small Greek indices λ, μ, ν … are coordinate indices and also always run from one to four. Frame indices are raised and lowered with the constant tensor g_{ab}.

We associate to every generalized spacetime

- A Lorentzian metric g determined by $g(E_a, E_b) = g_{ab}$.
- A connection ∇ specified by

$$g\big(\nabla_{E_a} E_b\,,\, E_c\big) = \Gamma_{abc} \qquad \text{or, equivalently,} \qquad \nabla_{E_a} E_b = \Gamma_{ab}{}^c E_c$$

  where $\Gamma_{ab}{}^c = g^{cd}\Gamma_{abd}$. The antisymmetry of Γ_{abc} in the last two indices, expresses the property that the connection ∇ is compatible with the metric.
- 24 connection torsion fields

$$T_{ab}{}^\mu \;=\; \Gamma_{ab}{}^c E_c{}^\mu - \Gamma_{ba}{}^c E_c{}^\mu - E_a\big(E_b{}^\mu\big) + E_b\big(E_a{}^\mu\big)$$

  They measure the deviation of ∇ from the Levi-Civita connection for the metric g. That is, $T_{ab}{}^\mu$ vanishes if and only if

$$\Gamma_{abc} \;=\; \tfrac{1}{2}\Big(-g\big(E_a, [E_b, E_c]\big) + g\big(E_c, [E_a, E_b]\big) + g\big(E_b, [E_c, E_a]\big)\Big)$$

- 36 curvature torsion fields

$$\begin{aligned} U_{k\ell ab} \;&= E_a\big(\Gamma_{b\ell k}\big) - E_b\big(\Gamma_{a\ell k}\big) \\ &\quad + \Gamma_{b\ell}{}^m\Gamma_{amk} - \Gamma_{a\ell}{}^m\Gamma_{b\,m\,k} - \big(\Gamma_{a\,b}{}^m - \Gamma_{ba}{}^m\big)\Gamma_{m\ell k} \;-\; W_{k\ell ab} \end{aligned}$$

  The curvature tensor $R^k{}_{\ell ab}$ for the connection ∇ is given by

$$\begin{aligned} R^k{}_{\ell ab} \;&=\; E_a\big(\Gamma_{b\ell}{}^k\big) - E_b\big(\Gamma_{a\ell}{}^k\big) \\ &\qquad + \Gamma_{b\ell}{}^m\Gamma_{am}{}^k - \Gamma_{a\ell}{}^m\Gamma_{bm}{}^k - \big(\Gamma_{ab}{}^m - \Gamma_{ba}{}^m - T_{ab}{}^m\big)\Gamma_{m\ell}{}^k \\ &=\; T_{ab}{}^m\Gamma_{m\ell}{}^k + U^k{}_{\ell ab} + W^k{}_{\ell ab} \end{aligned} \tag{B.3}$$

  where $T_{ab}{}^m E^\mu_m = T_{ab}{}^\mu$. In the event that $T_{ab}{}^\mu$ vanishes, the curvature torsion fields $U_{k\ell ab}$ measure the deviation of the Riemann curvature for the Levi - Civita connection from the Weyl tensor $W_{k\ell ab}$.
  The curvature tensor $R_{k\ell ab}$ for the connection ∇ has the symmetries

$$R_{k\ell ab} = -R_{\ell kab} \qquad R_{k\ell ab} = -R_{k\ell ba}$$

  Warning. The customary pair exchange symmetry and cyclic identity do not necessarily hold when the torsion $T_{ab}{}^\mu$ is not zero.
- 16 Bianchi fields

$$V_{abijk} \;=\; \nabla_i W_{abjk} + \nabla_k W_{abij} + \nabla_j W_{abki}$$

  or, equivalently, contracted Bianchi fields

$$V_{bjk} \;=\; g^{ai} V_{abijk} \;=\; \nabla^a W_{abjk}$$

  For the definition of $\nabla_i W_{abjk}$, see (B.4) below. The fields V_{abijk} vanish if and only if the Weyl tensor W_{abki} satisfies the Bianchi identities with respect to the connection ∇. Similarly, the fields V_{bjk} vanish if and only if the Weyl tensor satisfies the contracted Bianchi identities with respect to the connection ∇.

Remark B.1. The contracted Bianchi fields are equivalent to the Bianchi fields. This fact is an immediate consequence of the following algebraic identity. Suppose, A_{abijk} is antisymmetric in the first two indices and totally antisymmetric in the last three. Set

$$A_{aij} \;=\; g^{bk} A_{abijk} \qquad A^{\sharp}_{ak\ell} \;=\; \tfrac{1}{2}\,\epsilon_{k\ell ij} A_a{}^{ij}$$

Then $A_{abijk} = \frac{1}{2}\,\epsilon_{ijk}{}^{\ell}\big(A^{\sharp}_{a\ell b} - A^{\sharp}_{b\ell a} + A^{\sharp}_{\ell ab}\big)$.

Convention B.2. We define E, Γ, W, T, U, V to be covariant tensors (vector field valued in the case of E and T) on $\mathcal{U}$ whose components with respect to the **fixed** frame E_a are given by

$$\begin{aligned} E_a &= E_a{}^{\mu}\tfrac{\partial}{\partial x^{\mu}} & T(E_a,E_b) &= T_{ab}{}^{\mu}\tfrac{\partial}{\partial x^{\mu}} \\ \Gamma(E_a,E_b,E_c) &= \Gamma_{abc} & U(E_a,E_b,E_c,E_d) &= U_{abcd} \\ W(E_a,E_b,E_c,E_d) &= W_{abcd} & V(E_a,E_b,E_c) &= V_{abc} \end{aligned}$$

From this perspective:

- If X_1, X_2, X_3 are vector fields on $\mathcal{U}$, then $\Gamma(X_1,X_2,X_3) \;=\; X_1^a\,X_2^b\,X_3^c\,\Gamma_{abc}$. Here $X_i = X_i^a E_a$ for $i = 1,2,3$. In general, $\Gamma(X_1,X_2,X_3) \neq g(\nabla_{X_1}X_2, X_3)$.
- Covariant derivatives of all these tensors are well defined. For example,

$$\nabla_i W_{abcd} = E_i(W_{abcd}) - \Gamma_{ia}{}^{m}W_{mbcd} - \Gamma_{ib}{}^{m}W_{amcd} - \Gamma_{ic}{}^{m}W_{abmd} - \Gamma_{id}{}^{m}W_{abcm} \tag{B.4}$$

Definition B.2. *We refer to the system*

$$\big(\,T,\,U,\,V\,\big) \;=\; 0 \tag{B.5}$$

as the **generalized vacuum field equations***.*

Proposition B.1. *A generalized spacetime $E_a{}^{\mu}$, Γ_{abc}, $W_{abk\ell}$ on an open subset $\mathcal{U}$ of $\mathbb{R}^4$ is a solution to the generalized vacuum field equations* (B.5) *if and only if ∇ is the Levi-Civita connection for the metric g, and the associated Riemann curvature tensor coincides with the Weyl tensor W. In this event, g is a solution to the vacuum Einstein equations* (B.1) *on $\mathcal{U}$.*

Proposition B.2. *The tensors T, U, V have the algebraic symmetries:*

$$T_{ab}{}^{\mu} = -T_{ba}{}^{\mu} \qquad\qquad V_{kab} = -V_{kba} \tag{B.6a}$$
$$U_{abk\ell} = -U_{bak\ell} \qquad\qquad V^{k}{}_{kb} = 0 \tag{B.6b}$$
$$U_{abk\ell} = -U_{ab\ell k} \qquad\qquad V_{abc} + V_{bca} + V_{cab} = 0 \tag{B.6c}$$

They satisfy the **generalized Bianchi equations**

$$\big(\,\mathfrak{T},\,\mathfrak{U},\,\mathfrak{V}\,\big) = 0 \tag{B.7}$$

where, by definition,

$$\mathfrak{T}_a{}^{\mu} \;=\; \epsilon_a{}^{ijk}\,\big(\,\widehat{\nabla}_i T_{jk}{}^{\mu} - U^{m}{}_{kij}\,E_m{}^{\mu} - T_{jk}{}^{\nu}\tfrac{\partial}{\partial x^{\nu}}\,E_i{}^{\mu}\,\big) \tag{B.8a}$$
$$\mathfrak{U}_{cab} \;=\; \epsilon_c{}^{ijk}\,\big(\,\nabla_i U_{abjk} - U^{m}{}_{ijk}\,\Gamma_{mab} + \tfrac{1}{3}V_{abijk} - T_{ij}{}^{\mu}\tfrac{\partial}{\partial x^{\mu}}\,\Gamma_{kab}\,\big) \tag{B.8b}$$
$$\begin{aligned}\mathfrak{V}_{jk} \;=\; &\nabla_b V^{b}{}_{jk} + U^{a}{}_{mab}\,W^{mb}{}_{jk} - \tfrac{1}{2}U_{mjab}\,W^{abm}{}_{k} \\ &+ \tfrac{1}{2}U_{mkab}\,W^{abm}{}_{j} - \tfrac{1}{2}T_{ab}{}^{\mu}\,\tfrac{\partial}{\partial x^{\mu}}W^{ab}{}_{jk}\end{aligned} \tag{B.8c}$$

Here ϵ_{abcd} is totally antisymmetric and $\epsilon_{1234} = -1$. Furthermore, $\widehat{\nabla}_i$ is the tensor derivation that acts on frame indices as ∇_i and ignores coordinate indices. Explicitly, $\widehat{\nabla}_i T_{jk}{}^{\mu} = E_i(T_{jk}{}^{\mu}) - \Gamma_{ij}{}^{m} T_{mk}{}^{\mu} - \Gamma_{ik}{}^{m} T_{jm}{}^{\mu}$.

Remark B.2. The generalized Bianchi equations (B.7) are *identities*: they hold for all generalized spacetimes. Both (B.5) and (B.7) are quadratically nonlinear. Each, has exactly one linear term. Respectively, $-W_{k\ell ab}$ and $-V_{abijk}$ in the equations $U = 0$ and $\mathfrak{U} = 0$. The only coordinate index appears in the $T = 0$ and $\mathfrak{T} = 0$ equations. Observe that, for fixed $E_a{}^{\mu}$, Γ_{abc}, W_{abjk}, the equations (B.7) are linear and homogeneous in $T_{ab}{}^{\mu}$, U_{abjk} and V_{abijk}.

Our goal is to construct physically interesting vacuum spacetimes. In this appendix we have traded in the 10 traditional metric tensor fields for 50 frame, connection and Weyl fields and an additional 76 connection torsion, curvature torsion and Bianchi fields. How can this formalism be of any practical use? Not only are there 126 fields, but both the generalized vacuum and Bianchi equations are overdetermined, since the tensor V vanishes whenever T and U both vanish.

Here is a rough outline of our strategy. Regard the frame $E_a{}^{\mu}$ and general connection Γ_{ijk} as vector fields with values in $\mathbb{R}^{16}$ and $\mathbb{R}^{24}$ respectively. We conceptualize abstract gauge conditions as fixed affine linear subspaces $\mathcal{E} \subset \mathbb{R}^{16}$ and $\mathcal{G} \subset \mathbb{R}^{24}$. The frame and connection are gauge fixed when $E_a{}^{\mu}(p) \in \mathcal{E}$ and $\Gamma_{ijk}(p) \in \mathcal{G}$ for all points $p \in \mathcal{U}$. No conditions are imposed on the Weyl tensor. There are

$$\dim \mathcal{E} + \dim \mathcal{G} + 10 \ \leq \ 50$$

independent gauge fixed frame, gauge fixed connection and Weyl fields. An abstract gauge fixed, generalized spacetime is summarized by a field Φ on $\mathcal{U}$ taking values in $\mathcal{E} \oplus \mathcal{G} \oplus \mathbb{R}^{10}$.

Abstract gauge conditions should have three properties:

- *Property B.1.* They are always "locally realizable". That is, near each point in every spacetime there is a coordinate system and a frame such that the components of the frame, with respect to the coordinate vector fields, and the components of the Levi - Civita connection, with respect to the frame, lie in the gauge subspaces $\mathcal{E}$ and $\mathcal{G}$. In other words, abstract gauge conditions should not exclude, a priori, any spacetimes.
- *Property B.2.* They are "symmetric hyperbolic". In the present context, a symmetric hyperbolic system of partial differential equations for the column vector v is a system

$$A^a \ E_a(v) \ = \ f$$

  where A^a is a symmetric matrix and $A^3 + A^4$ is strictly positive definite. Now, pick bases for $\mathcal{E}$ and $\mathcal{G}$, and rewrite the 60 equations $T = 0$ and $U = 0$ explicitly in terms of the components of the field Φ. It is required that one can select exactly $\dim \mathcal{E}$ linear combinations of the 24 connection torsion equations ($T = 0$) and exactly $\dim \mathcal{G}$ linear combinations of the 36 curvature torsion equations ($U = 0$) equations and 10 linear combinations of the 16 Bianchi equations ($V = 0$) which taken together comprise a (quasilinear) symmetric hyperbolic system for Φ which we will refer to as **(SHS)** .

Property B.2 is not just wishful thinking, as it may first appear. Only the principal parts of (B.5),

$$\begin{aligned}&E_a(E_b{}^\mu) - E_b(E_a{}^\mu)\\&E_j(\Gamma_{kba}) - E_k(\Gamma_{jab})\\&E_i(W_{abjk}) + E_k(W_{abij}) + E_j(W_{abki})\end{aligned}$$

have to be considered in the quest for symmetric hyperbolic equations. Furthermore, only the frame and connection fields in the principal parts have to be written out in terms of Φ. It is unnecessary to open up and look at the occurrences of the frame fields inside the first order differential operators E_a. At this level, it is required that there are, in turn, linear combinations of the principal parts that are symmetric hyperbolic. In principle, the field Φ, that contains all information about the generalized spacetime, is now uniquely determined, given appropriate data, by **(SHS)** . However, there is an important catch. **(SHS)** and the abstract gauge conditions imply that some part of the tensors T, U and V vanish, but not all. The remaining components are summarized in the *constraint field* $\Phi^\sharp$. If **(SHS)** is satisfied and $\Phi^\sharp$ vanishes, then $E_a{}^\mu$, Γ_{abc}, W_{abjk} is a solution to the generalized vacuum field equations.

- *Property B.3.* They are "dual symmetric hyperbolic". If **(SHS)** is satisfied, it is required that, in an entirely similar way, judicious linear combinations of the generalized Bianchi equations (B.7) can be brought into the form of a *linear, homogeneous* symmetric hyperbolic system for $\Phi^\sharp$ which we refer to as $\widehat{\textbf{(SHS)}}$. In particular, if the data for any well posed problem for the system $\widehat{\textbf{(SHS)}}$ vanishes, then the constraint field $\Phi^\sharp$ vanishes everywhere.

It is much simpler to carry out this general methodology in practice than to formulate it in broad conceptual terms. Different problems require different gauges and symmetric hyperbolic systems. In Appendix C, we introduce the wavefront gauge for Lorentzian manifolds. In Appendix D, we fix the abstract wavefront gauge and select symmetric hyperbolic subsystems from the generalized vacuum and Bianchi equations that are particularly suited to the problem we are solving in this paper.

Proof (of the Generalized Bianchi Equations (B.7)*).*

- $\mathfrak{T} = 0$: Write $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. Repeatedly exploiting the total antisymmetry of ϵ^{aijk} and then using the Jacobi identity,

$$\epsilon^{aijk}\big(\nabla_i\nabla_j E_k - \nabla_j\nabla_i E_k - \nabla_{[E_i,E_j]}E_k\big) \\ = \epsilon^{aijk}\Big(\nabla_i\big(T(E_j,E_k)\big) + T\big(E_i,[E_j,E_k]\big)\Big)$$

It follows that $\epsilon^{aijk}R^b{}_{ijk}E_b^\mu = \epsilon^{aijk}\big(\widehat{\nabla}_i T_{jk}{}^\mu + T_{jk}{}^b\Gamma_{bi}{}^c E_c{}^\mu - T_{jk}{}^\nu\frac{\partial}{\partial x^\nu}E_i{}^\mu\big)$. Substituting (B.3) for the curvature $R^b{}_{ijk}$, the identity $\mathfrak{T} = 0$ follows.

- $\mathfrak{U} = 0$: Apply the operator $\epsilon^{aijk}\nabla_i$ to the identity (B.3) in the form

$$R_{bcjk} \;=\; U_{bcjk} + T_{jk}{}^m\Gamma_{mcb} + W_{bcjk}$$

and use the standard Bianchi identity for the curvature tensor corresponding to a connection with torsion, $\epsilon^{aijk}\big(\nabla_i R_{bcjk} + T_{ij}{}^\ell R_{bc\ell k}\big) = 0$.

- $\mathfrak{V} = 0$: The divergence $\nabla_b V^b{}_{jk} = -\frac{1}{2}\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)W^{ab}{}_{jk}$. Express the commutators in terms of the curvature tensor:

$$\begin{aligned}&(\nabla_c\nabla_d - \nabla_d\nabla_c)W_{abjk}\\ &= -T_{cd}{}^{\ell}\nabla_\ell W_{abjk} - R^{\ell}{}_{acd}W_{\ell bjk} - R^{\ell}{}_{bcd}W_{a\ell jk} - R^{\ell}{}_{jcd}W_{ab\ell k} - R^{\ell}{}_{kcd}W_{abj\ell}\end{aligned}$$

Substitute (B.3) for the curvature, contract indices, write out the covariant derivatives, rearrange, collect terms and cancel to obtain $\mathfrak{V} = 0$. $\square$

C. The Wavefront Gauge for Lorentzian Manifolds

Here, we introduce the *geometric wavefront gauge* in the language of Lorentzian geometry. The *abstract wavefront gauge*, in the language of generalized spacetimes, is introduced in Appendix D.

Proposition C.1 (Geometric wavefront gauge). *Every point on any Lorentzian 4-manifold (M, g) has an open neighborhood on which there are coordinates*

$$(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$$

and an oriented frame (E_1, E_2, E_3, E_4) such that $g(E_a,\, E_b) = g_{ab}$, see (B.2)*, such that E_3 and E_4 are both future directed vector fields, and such that*

(a) the coordinate functions u and $\underline{u}$ are solutions to the eikonal equation, that is, $g^{ab}E_a(u)E_b(u) = 0$ and $g^{ab}E_a(\underline{u})E_b(\underline{u}) = 0$.
(b) the vector field E_4 is minus the gradient of u, that is $g(E_4, \,\cdot\,) = -\mathrm{d}u$.
(c) the coordinates ξ^1, ξ^2 are constant along the integral curves of E_4.
(d) the function $e_3 = E_4(\underline{u})$ is strictly positive and the vector field $e_3\, E_3$ is minus the gradient of $\underline{u}$, that is $g(e_3 E_3, \,\cdot\,) = -\mathrm{d}\underline{u}$.
(e) E_4 and $e_3 E_3$ are null geodesic vector fields.
(f) the frame vector fields E_1 and E_2 satisfy

$$g\big(\nabla_{E_4} E_1,\, E_2\big) \,=\, 0$$

where ∇ is the Levi-Civita connection.

Proof (Informal). To start with, suppose M is Minkowski space. Let $X = (X^0, \mathbf{X}) = (X^0, X^1, X^2, X^3) \in \mathbb{R} \times \mathbb{R}^3$ be standard Cartesian coordinates. For every fixed $\epsilon > 0$, define $u, \underline{u} : \{X \in \mathbb{R}^4 :\ X^0 > -\epsilon\} \to \mathbb{R}$ by

$$u(p) = \sup_{(-\epsilon, \mathbf{X}) \in I^-(p)} X^1 \quad , \quad \underline{u}(p) = -\inf_{(-\epsilon, \mathbf{X}) \in I^-(p)} X^1 .$$

Here, $I^-(p)$ $\big($resp., $I^+(p)\big)$ is the chronological past (future) of p, that is, all points that can be reached from p by traveling along past (future) directed, piecewise smooth, timelike curves. Note that

- by construction, $u(p) \le u(q)$ and $\underline{u}(p) \le \underline{u}(q)$ for all $q \in I^+(p)$,
- there is a sufficiently small $\epsilon > 0$ and an open neighborhood $U \subset M$ of the origin $X = 0$, on which u, $\underline{u}$ are smooth and $\mathrm{d}u$, $\mathrm{d}\underline{u}$ are linearly independent.

It follows that, $\mathrm{d}u$ and $\mathrm{d}\underline{u}$ are timelike or null on U.

Suppose $\mathrm{d}u$ is timelike at a point $p \in U$. Then, the level set Σ of u passing through p would be (locally) a smooth spacelike hypersurface. If $q \in I^+(p)$ is sufficiently close to p, then every past directed timelike curve from q intersects Σ, and

$$u(q) = \sup_{(-\epsilon,\mathbf{X})\in I^-(q)} X^1 = \sup_{p'\in\Sigma\cap I^-(q)} \sup_{(-\epsilon,\mathbf{X})\in I^-(p')} X^1 = \sup_{p'\in\Sigma\cap I^-(q)} u(p') = u(p).$$

because $I^-(q)\cap H = \bigcup_{p'\in\Sigma\cap I^-(q)} I^-(p')\cap H$, with $H = \{X \in \mathbb{R}^4 :\ X^0 = -\epsilon\}$. This contradicts the assumption that $\mathrm{d}u(p)$ is timelike. Therefore, $\mathrm{d}u$ is null. Similarly, $\mathrm{d}\underline{u}$ is null.

Now, fix a point p_0 on any Lorentzian manifold M, and let $(X^0,\ \mathbf{X})$ be smooth local coordinates that vanish at p, with $-\mathrm{d}X^0$ a future directed 1-form. Precisely the same construction for u and $\underline{u}$ works on a suitably small neighborhood $U \subset M$ of p_0.

Define vector fields L and $\underline{L}$ by $g(L,\,\cdot\,) = -\mathrm{d}u$ and $g(\underline{L},\,\cdot\,) = -\mathrm{d}\underline{u}$. They are future null. Define $E_4 = L$, $e_3 = L(\underline{u}) > 0$ and $E_3 = e_3^{-1}\underline{L}$. In particular, $g(E_3, E_4) = -1$.

Condition (e) is equivalent to $\nabla_L L = 0$ and $\nabla_{\underline{L}}\underline{L} = 0$. These are consequences of the general fact that for any function w, the acceleration $\nabla_W W$ of its gradient field W is the gradient field of the function $\frac{1}{2}g(W, W)$.

Let K_1 and K_2 be spacelike, orthonormal vector fields, perpendicular to E_3 and E_4. Define $\left(\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)\left(\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right)$ where α satisfies the differential equation $E_4(\alpha) = -\,g\big(\nabla_{E_4}K_1,\ K_2\big)$ along the integral curves of E_4. $\square$

Remark C.1. The spacelike vector fields E_1 and E_2 and the null geodesic vector field E_4 are tangent to each level set of u. Each level set of u is a union of null geodesics, the lines of constant ξ^1, ξ^2 and u.
Similarly, E_1, E_2 and the null geodesic vector field $e_3\,E_3$ are tangent to each level set of $\underline{u}$. Each level set of $\underline{u}$ is a union of null geodesics. They are, in general, not given by the lines of constant ξ^1, ξ^2 and $\underline{u}$.
We refer to the intersections of level sets of u and $\underline{u}$ as *wavefronts*. They are spacelike and their tangent space is spanned by E_1 and E_2.

Proposition C.2. *Fix a coordinate system and frame as in the geometric wavefront gauge of Proposition C.1. Let ${E_a}^\mu$ be the components of the frame field with respect to the coordinate system and let Γ_{abc} be the components of the Levi-Civita connection ∇,*

$$E_a = {E_a}^\mu\,\frac{\partial}{\partial x^\mu}, \qquad \Gamma_{abc} = g\big(\nabla_{E_a}E_b,\ E_c\big).$$

Then

$$\Big({E_a}^\mu\Big) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 1 \\ 0 & 0 & e_3 & 0 \end{pmatrix}$$

where $*$ is a generic symbol. Moreover,

$$\Big(\Gamma_{a(bc)}\Big) = \begin{pmatrix} * & \underline{\chi}_{11} & \underline{\chi}_{12} & \chi_{11} & \chi_{12} & \bigstar \\ * & \underline{\chi}_{21} & \underline{\chi}_{22} & \chi_{21} & \chi_{22} & \bigstar \\ * & 0 & 0 & \bigstar & \bigstar & * \\ 0 & * & * & 0 & 0 & 0 \end{pmatrix}$$

where the matrix indices (bc) *run over the ordered sequence*

$$(12), \quad (31), \quad (32), \quad (41), \quad (42), \quad (34).$$

Here, χ_{AB} *and* $\underline{\chi}_{AB}$ *are the second fundamental forms of the wavefronts in the normal null directions* E_3 *and* E_4*. As such,*

$$\chi_{AB} = \chi_{BA} \qquad \underline{\chi}_{AB} = \underline{\chi}_{BA}.$$

Moreover, the entries filled with the generic symbol $\bigstar$ *satisfy*

$$\Gamma_{A34} = \Gamma_{3A4} \qquad A = 1, 2.$$

Proof. We first verify the 0's and 1's in these matrices. The entries of $(E_a{}^\mu)$ follow directly from Proposition C.1, for example, $E_3(u) = \mathrm{d}u(E_3) = -g(E_4, E_3) = 1$ by (b). The zeros in (Γ_{abc}) are accounted for by (f), by $\nabla_{E_4} E_4 = 0$, see (e), and by the fact that $\nabla_{E_3} E_3$ is proportional to E_3, see (e). Next, $\Gamma_{123} - \Gamma_{213} = 0$ ($\underline{\chi}_{12} = \underline{\chi}_{21}$) and $\Gamma_{124} - \Gamma_{214} = 0$ ($\chi_{12} = \chi_{21}$) follow, by (d) and (b), from $[E_1, E_2](\underline{u}) = 0$ and $[E_1, E_2](u) = 0$, respectively. Finally, $\Gamma_{A34} - \Gamma_{3A4} = 0$ follows from $[E_A, E_3](u) = 0$. $\square$

D. The Abstract Wavefront Gauge

In this Appendix, we leave the realm of Lorentz 4-manifolds, and speak exclusively in the language of generalized spacetimes, as in Appendix B.

We now define the *abstract wavefront gauge* and show that it has the Properties B.1 through B.3.

It is convenient to introduce the complex frame

$$(F_1, F_2, F_3, F_4) = (D, \overline{D}, N, L), \qquad F_a = \mathbf{F}_a{}^\mu \tfrac{\partial}{\partial x^\mu}$$

where

$$D = 2^{-\frac{1}{2}}(E_1 + iE_2), \qquad \overline{D} = 2^{-\frac{1}{2}}(E_1 - iE_2), \qquad N = E_3, \qquad L = E_4.$$

These fields are sections of the complexified tangent bundle.

Convention D.1. Let t be the constant matrix for which $\mathbf{F}_a{}^\mu = t_a{}^b E_b{}^\mu$, and t^{-1} its inverse. For every tensor field S in Appendix B with components $S_{a_1 \dots a_n}{}^{b_1 \dots b_m}$ with respect to the real frame (E_a), the corresponding components with respect to the complex frame (F_a) are distinguished by boldface letters and defined by

$$\mathbf{S}_{a_1 \dots a_n}{}^{b_1 \dots b_m} = t_{a_1}{}^{i_1} \cdots t_{a_n}{}^{i_n} S_{i_1 \dots i_n}{}^{j_1 \dots j_m} (t^{-1})_{j_1}{}^{b_1} \cdots (t^{-1})_{j_m}{}^{b_m} \tag{D.1}$$

If the tensor field S also carries coordinate indices, they are unaffected by (D.1). An equivalent statement to (D.1) is: if S is the tensor field for which

$$S(E_{a_1}, \dots, E_{a_n}) = S_{a_1 \dots a_n}{}^{b_1 \dots b_m} E_{b_1} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} E_{b_m}$$

and S is extended complex linearly in its n arguments, then

$$S(F_{a_1}, \dots, F_{a_n}) = \mathbf{S}_{a_1 \dots a_n}{}^{b_1 \dots b_m} F_{b_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} F_{b_m}$$

The transformation (D.1) commutes with contraction of indices. Accordingly, the indices of boldface fields have to be raised and lowered with

$$\mathbf{g}^{ab} = g^{ij}(t^{-1})_i{}^a(t^{-1})_j{}^b \qquad \mathbf{g}_{ab} = t_a{}^i t_b{}^j g_{ij}$$

see (B.2) and (2.3). The complex components $\mathbf{S}_{a_1\ldots a_n}{}^{b_1\ldots b_m}$ are only introduced for notational convenience. The corresponding tensor field S will, however, always be real, in the sense that if all its n arguments are real, then the result is real.

For the particular covariant tensors E, Γ, W, T, U, V (see, Convention B.2), the transformation (D.1) becomes

$$\begin{aligned} \mathbf{F}_a{}^\mu \frac{\partial}{\partial x^\mu} &= F_a & \mathbf{T}_{ab}{}^\mu \frac{\partial}{\partial x^\mu} &= T(F_a, F_b) \\ \boldsymbol{\Gamma}_{abc} &= \Gamma(F_a, F_b, F_c) & \mathbf{U}_{abcd} &= U(F_a, F_b, F_c, F_d) \\ \mathbf{W}_{abcd} &= W(F_a, F_b, F_c, F_d) & \mathbf{V}_{abc} &= V(F_a, F_b, F_c) \end{aligned}$$

Definition D.1. *Let*

$$\Phi \;=\; (e,\, \gamma,\, w):\;\; \mathcal{U} \;\to\; \mathcal{R} \;\subset\; \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 \qquad \textit{see (2.1)}$$

be a sufficiently differentiable field satisfying conditions $(\star)$ *and* $(\star\star)$ *in Definition 2.1. The* **Abstract Wavefront Gauge Spacetime**

$$M_\Phi \;=\; (\, \mathbf{F}_a{}^\mu,\; \mathbf{\Gamma}_{abc},\; \mathbf{W}_{abcd}\,)$$

is defined just as in Definition 2.1.

Remark D.1. Definition D.1 implicitly fixes abstract gauge conditions in the sense of Appendix B. The affine spaces $\mathcal{E}$ and $\mathcal{G}$ have real dimensions, respectively, 7 and 14. That is, the field Φ has 31 real components.

Remark D.2. Observe that Proposition C.2 is the statement that the abstract wavefront gauge in Definition D.1 is locally realizable in the sense of Appendix B (Property B.1).

Proposition D.1. *Let* M_Φ *be an abstract wavefront gauge spacetime. Let* λ_1, λ_2, λ_3, λ_4 *be strictly positive weight functions on* $\mathcal{U}$. *Then, there are unique fields*

$$\begin{aligned} (\mathfrak{t}, \mathfrak{u}, \mathfrak{v}):& \quad \mathcal{U} \to\; \mathcal{R} \;\subset\; \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 && \textit{see (2.1)} \\ (t, u, v):& \quad \mathcal{U} \to\; \widehat{\mathcal{R}} \;\subset\; \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3 && \textit{see (2.6)} \end{aligned}$$

and such that

$$\left(\mathbf{T}_{(ab)}{}^\mu\right) = \begin{pmatrix} i\,t_1 & i\,t_2 & 0 & 0 \\ \overline{t}_4 & \overline{t}_5 & 0 & 0 \\ t_4 & t_5 & 0 & 0 \\ -\,\mathfrak{t}_1 & -\,\mathfrak{t}_2 & t_3 & 0 \\ -\,\overline{\mathfrak{t}}_1 & -\,\overline{\mathfrak{t}}_2 & \overline{t}_3 & 0 \\ \mathfrak{t}_4 & \mathfrak{t}_5 & -\,\mathfrak{t}_3 & 0 \end{pmatrix}$$

$$\left(\mathbf{U}_{(ab)(jk)}\right) = \begin{pmatrix} u_2 + \overline{u}_2 & u_7 - \overline{u}_8 & u_8 - \overline{u}_7 & -\mathfrak{u}_3 - \overline{\mathfrak{u}}_4 & \mathfrak{u}_4 + \overline{\mathfrak{u}}_3 & \mathfrak{u}_8 - \overline{\mathfrak{u}}_8 \\ \overline{u}_9 & -\overline{\mathfrak{u}}_7 & -\mathfrak{u}_6 & \overline{u}_5 & -\overline{u}_6 & -\overline{\mathfrak{u}}_5 \\ -u_9 & -\mathfrak{u}_6 & -\mathfrak{u}_7 & -u_6 & u_5 & -\mathfrak{u}_5 \\ -u_1 & u_4 & -u_3 & -\mathfrak{u}_1 & -\mathfrak{u}_2 & -\mathfrak{u}_3 + \overline{\mathfrak{u}}_4 \\ \overline{u}_1 & -\overline{u}_3 & \overline{u}_4 & -\mathfrak{u}_2 & -\overline{\mathfrak{u}}_1 & \mathfrak{u}_4 - \overline{\mathfrak{u}}_3 \\ u_2 - \overline{u}_2 & u_7 + \overline{u}_8 & u_8 + \overline{u}_7 & -\mathfrak{u}_3 + \overline{\mathfrak{u}}_4 & \mathfrak{u}_4 - \overline{\mathfrak{u}}_3 & \mathfrak{u}_8 + \overline{\mathfrak{u}}_8 \end{pmatrix}$$

$$
\Big(\mathbf{V}_{a(jk)}\Big) = \left(\begin{matrix} \frac{1}{\lambda_2}(-\mathfrak{v}_2 + v_1) - \frac{1}{\lambda_3}\overline{v}_3 & -\frac{1}{\lambda_4}\overline{\mathfrak{v}}_5 & \frac{1}{\lambda_3}(\mathfrak{v}_3 - v_2) \\ \frac{1}{\lambda_2}(\overline{\mathfrak{v}}_2 - \overline{v}_1) + \frac{1}{\lambda_3}v_3 & \frac{1}{\lambda_3}(\overline{\mathfrak{v}}_3 - \overline{v}_2) & -\frac{1}{\lambda_4}\mathfrak{v}_5 \\ \frac{1}{\lambda_3}(\mathfrak{v}_3 - \overline{\mathfrak{v}}_3 - v_2 + \overline{v}_2) & \frac{1}{\lambda_4}(-\overline{\mathfrak{v}}_4 + \overline{v}_3) & \frac{1}{\lambda_4}(-\mathfrak{v}_4 + v_3) \\ \frac{1}{\lambda_2}(-v_2 + \overline{v}_2) & \frac{1}{\lambda_3}\overline{v}_3 & \frac{1}{\lambda_3}v_3 \end{matrix}\right.
$$

$$
\left.\begin{matrix} \cdots & -\frac{1}{\lambda_1}\mathfrak{v}_1 & \frac{1}{\lambda_2}\overline{v}_2 & \frac{1}{\lambda_2}(-\mathfrak{v}_2 + v_1) + \frac{1}{\lambda_3}\overline{v}_3 \\ \cdots & \frac{1}{\lambda_2}v_2 & -\frac{1}{\lambda_1}\overline{\mathfrak{v}}_1 & \frac{1}{\lambda_2}(-\overline{\mathfrak{v}}_2 + \overline{v}_1) + \frac{1}{\lambda_3}v_3 \\ \cdots & \frac{1}{\lambda_2}(\mathfrak{v}_2 - v_1) & \frac{1}{\lambda_2}(\overline{\mathfrak{v}}_2 - \overline{v}_1) & \frac{1}{\lambda_3}(\mathfrak{v}_3 + \overline{\mathfrak{v}}_3 - v_2 - \overline{v}_2) \\ \cdots & -\frac{1}{\lambda_1}v_1 & -\frac{1}{\lambda_1}\overline{v}_1 & -\frac{1}{\lambda_2}(v_2 + \overline{v}_2) \end{matrix}\right)
$$

The matrix indices (ab), (jk) run over the ordered sequence

$$
(12) \quad (31) \quad (32) \quad (41) \quad (42) \quad (34)
$$

Proof. In general, the tensors T, U and V lie (pointwise) in spaces of real dimension 24, 36 and 16, respectively (see, equations (B.6)). By direct inspection, the equations

$$
\begin{matrix} \mathbf{T}_{12}{}^{3} = 0 & \mathbf{T}_{31}{}^{3} = 0 & \mathbf{T}_{ab}{}^{4} = 0 \\ \mathbf{U}_{3441} = \mathbf{U}_{4134} & \Im\,\mathbf{U}_{3132} = 0 & \Im\,\mathbf{U}_{4142} = 0 \end{matrix}
$$

hold for every field Φ, and, consequently, the associated tensors T, U and V lie in subspaces of real dimension $24 - 9 = 15$, $36 - 4 = 32$ and $16 - 0 = 16$, respectively. By construction, the matrices on the right hand sides of the equations above lie in these subspaces. The linear map from $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \oplus (t, u, v)$ to these matrices has maximal rank 63, which is $15+32+16$. Therefore, the fields $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and (t, u, v) exist and are unique. □

Proposition D.1 defines a unique "splitting" of the nonzero components of (T, U, V) into two sets, $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and (t, u, v). This splitting and the role of the weight functions λ_i is clarified by the following proposition and the subsequent remarks.

Proposition D.2. *For every choice of strictly positive weight functions λ_1, λ_2, λ_3, λ_4 on $\mathcal{U}$, the system of equations*

$$
(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0
$$

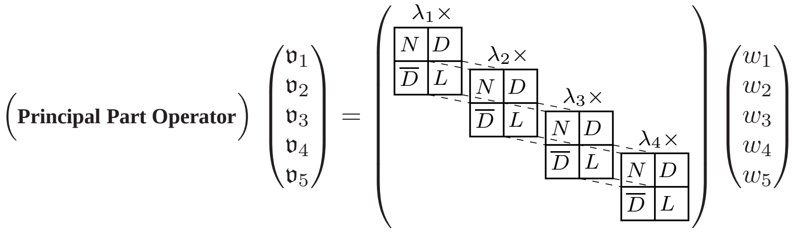
is equivalent to the system **(SHS)** , $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$*, in Definition 2.3, which is a (quasi-linear) symmetric hyperbolic system for the field $\Phi = (e, \gamma, w)$ provided $e_3 > 0$. In particular, the abstract wavefront gauge of Definition D.1 has Property B.2.*

Proof. The equivalence of $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$ with the symmetric hyperbolic system **(SHS)** in Definition 2.3 is by direct (machine) verification. □

Remark D.3. The term "symmetric" is a slight misnomer, in the sense that the matrices $\mathbf{A}^{\mu}(\Phi)$ determining the principal part $\mathbf{A}^{\mu}(\Phi)\frac{\partial}{\partial x^{\mu}}$ are complex Hermitian rather than real symmetric.

Remark D.4. **(SHS)** is of a form which is particularly suited to constructing solutions. The reason is that the first two blocks $\mathbf{A}_1(\Phi)$, $\mathbf{A}_2(\Phi)$ of the principal part, corresponding to the principal parts of $\mathfrak{t} = 0$ and $\mathfrak{u} = 0$, are diagonal and only L or N appear.

Remark D.5. Note that



The dotted lines in the schematic diagram for the 5×5 matrix on the right hand side indicate that the overlapping entries are the sums $\lambda_i L + \lambda_{i+1} N$. Each $\left(\begin{smallmatrix} N & D \\ \overline{D} & L \end{smallmatrix}\right)$ block is symmetric hyperbolic, and consequently, so is the 5×5 matrix for any choice of the strictly positive weight functions.

Remark D.6. The weights have a natural interpretation in terms of energies. The energy current naturally associated to **(SHS)** , $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$, is the vector field $j^\mu = \Phi^\dagger \mathbf{A}^\mu(\Phi)\,\Phi$. Estimates are obtained by applying the divergence theorem to

$$\partial_\mu j^\mu = \Phi^\dagger (\partial_\mu \mathbf{A}^\mu)\Phi + 2\Re\big(\Phi^\dagger \mathbf{f}(\Phi)\big).$$

The energies are integrals over the spacelike components of the boundary. The functions λ_1, λ_2, λ_3, λ_4 appear in the boundary integrals and play the role of weights for the components w_1, w_2, w_3, w_4, w_5.

Proposition D.3. *Let λ_1, λ_2, λ_3, λ_4 be strictly positive weight functions on $\mathcal{U}$. The components of (t, u, v) are given by* (2.7) *in Definition 2.4. The field $\Phi^\sharp = (t, u, v)$: $\mathcal{U} \to \widehat{\mathcal{R}}$ is called the* **constraint field** *associated to $\Phi = (e, \gamma, w)$.*

Proof. By direct (machine) calculation. □

The generalized vacuum field equations (B.5) reduce, in the abstract wavefront gauge, to **(SHS)** and $\Phi^\sharp = 0$, see Proposition D.1. How can we ensure that a solution to **(SHS)** also satisfies $\Phi^\sharp = 0$? The answer is given in Proposition D.4.

Proposition D.4. *Assume that $\Phi = (e, \gamma, w)$ satisfies $e_3 > 0$ and solves* **(SHS)** *or, equivalently, $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$. Let the Latin indices in ${\mathfrak{T}_a}^\mu$, $\mathfrak{U}_{abc}$, $\mathfrak{V}_{ab}$ denote components of the fields* (B.8) *with respect to the complex frame field F_a (see, Convention D.1). The subsystem of the generalized Bianchi equations* (B.7) *given by*

$$\begin{pmatrix} {\mathfrak{T}_4}^1 \\ {\mathfrak{T}_4}^2 \\ {\mathfrak{T}_1}^3 \\ {\mathfrak{T}_2}^1 \\ {\mathfrak{T}_2}^2 \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{U}_{414} \\ \frac{1}{2}(\mathfrak{U}_{412} + \mathfrak{U}_{434}) \\ \mathfrak{U}_{214} \\ \mathfrak{U}_{114} \\ \mathfrak{U}_{223} \\ \mathfrak{U}_{123} \\ \frac{1}{2}(\mathfrak{U}_{112} + \mathfrak{U}_{134}) \\ \frac{1}{2}(\mathfrak{U}_{212} + \mathfrak{U}_{234}) \\ \mathfrak{U}_{332} \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{V}_{41} \\ \frac{1}{2}(\mathfrak{V}_{12} + \mathfrak{V}_{34}) \\ \mathfrak{V}_{23} \end{pmatrix} = 0 \tag{D.2}$$

is equivalent to the system $\widehat{\textbf{(SHS)}}$, $\widehat{\mathbf{A}}(\Phi)\Phi^\sharp = \widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\Phi^\sharp$, *in Proposition 2.3, which is a linear homogeneous symmetric hyperbolic system for $\Phi^\sharp = (t, u, v)$. In particular, the abstract wavefront gauge of Definition D.1 has Property B.3.*

Proof. By assumption, $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$. The equivalence of (D.2) with the linear, homogeneous symmetric hyperbolic system $\widehat{\textbf{(SHS)}}$ is by direct (machine) verification. □

E. Symmetries: Proofs

In this section, we prove that the field transformations introduced in Section 3 are field symmetries (see, Definition 3.1).

Recall from Section 3 the definition of a field transformation S. In this Appendix we take a slightly different perspective and regard $x \in \mathcal{U}$ and $x' \in \mathcal{U}'$ as two sets of (global) coordinates on the same 4-dimensional manifold. Similarly, we regard Φ, Λ, F_a, ∇, W and their primed counterparts as objects on this 4-manifold. Here, ∇ is the connection associated with Γ, and W is viewed as a 4-covariant tensor (see, Appendix B and Convention D.1). Field transformations are defined, in this Appendix, through their action on the coordinates x, the complex frame vector fields F_a, the connection ∇, the 4-covariant tensor W, and the strictly positive weight functions Λ. The definitions of $\mathfrak{J}$, $\mathfrak{A}$ given below are equivalent to the corresponding definitions in Section 3. Those of $\mathfrak{C}$, $\mathfrak{Z}$ are slight generalizations, because $\mathfrak{C}^1$, $\mathfrak{C}^2$ and ζ are allowed to depend on (x^1, x^2, x^4).

Convention E.1. For the rest of this appendix, it is implicitly assumed that $x' = S \cdot x$ is a local diffeomorphism on $\mathbb{R}^4$. With this understanding, it is unnecessary to specify the ranges of the x and x', because the discussion is purely algebraic. A dot, $\cdot$, always denotes a group action.

Angular coordinate transformation $\mathfrak{C}$. Let $\mathfrak{C}^1(x^1, x^2, x^4)$, $\mathfrak{C}^2(x^1, x^2, x^4)$ be functions.

$$\begin{aligned} x' = \mathfrak{C} \cdot x &= \big(\mathfrak{C}^1(x^1, x^2, x^4),\ \mathfrak{C}^2(x^1, x^2, x^4),\ x^3,\ x^4\big) & \mathfrak{C} \cdot \nabla &= \nabla \\ \big(\mathfrak{C} \cdot \mathbf{F}\big)_a{}^\mu \tfrac{\partial}{\partial (x')^\mu} &= \mathbf{F}_a{}^\mu \tfrac{\partial}{\partial x^\mu} & \mathfrak{C} \cdot W &= W \\ \mathfrak{C} \cdot \Lambda &= \Lambda \end{aligned}$$

$U(1)$ *transformation* $\mathfrak{Z}$. Let $\zeta = \zeta(x^1, x^2, x^4) \in U(1)$.

$$\begin{aligned} x' = \mathfrak{Z} \cdot x &= x & \mathfrak{Z} \cdot \nabla &= \nabla \\ \big(\mathfrak{Z} \cdot \mathbf{F}\big)_a{}^\mu \tfrac{\partial}{\partial (x')^\mu} &= \big(\zeta\, \mathbf{F}_1{}^\mu,\ \zeta^{-1}\, \mathbf{F}_2{}^\mu,\ \mathbf{F}_3{}^\mu,\ \mathbf{F}_4{}^\mu\big) \tfrac{\partial}{\partial x^\mu} & \mathfrak{Z} \cdot W &= W \\ \mathfrak{Z} \cdot \Lambda &= \Lambda \end{aligned}$$

Global Isotropic Scaling $\mathfrak{J}$. Let $\mathfrak{J} > 0$ be a constant.

$$\begin{aligned} x' = \mathfrak{J} \cdot x &= (x^1,\ x^2,\ \mathfrak{J}x^3,\ \mathfrak{J}x^4) & \mathfrak{J} \cdot \nabla &= \nabla \\ \big(\mathfrak{J} \cdot \mathbf{F}\big)_a{}^\mu \tfrac{\partial}{\partial (x')^\mu} &= \mathfrak{J}^{-1}\, \mathbf{F}_a{}^\mu \tfrac{\partial}{\partial x^\mu} & \mathfrak{J} \cdot W &= \mathfrak{J}^2 W \\ \mathfrak{J} \cdot \Lambda &= \Lambda \end{aligned}$$

Global Anisotropic Scaling $\mathfrak{A}$. Let $\mathfrak{A} \neq 0$ be a constant.

$$\begin{aligned} x' = \mathfrak{A} \cdot x &= (\tfrac{1}{\mathfrak{A}} x^1,\ \tfrac{1}{\mathfrak{A}} x^2,\ x^3,\ \mathfrak{A}^2 x^4) & \mathfrak{A} \cdot \nabla &= \nabla \\ \big(\mathfrak{A} \cdot \mathbf{F}\big)_a{}^\mu \tfrac{\partial}{\partial (x')^\mu} &= \big(\tfrac{1}{\mathfrak{A}}\, \mathbf{F}_1{}^\mu,\ \tfrac{1}{\mathfrak{A}}\, \mathbf{F}_2{}^\mu,\ \tfrac{1}{\mathfrak{A}^2}\, \mathbf{F}_3{}^\mu,\ \mathbf{F}_4{}^\mu\big) \tfrac{\partial}{\partial x^\mu} & \mathfrak{A} \cdot W &= \mathfrak{A}^2 W \\ \mathfrak{A} \cdot \Lambda &= \operatorname{diag}(1, \mathfrak{A}^2, \mathfrak{A}^4, \mathfrak{A}^6)\, \Lambda \end{aligned}$$

Remark E.1. The action on the frame induces a global conformal transformation of the associated metric: $\mathfrak{C} \cdot g = g,\ \mathfrak{Z} \cdot g = g,\ \mathfrak{J} \cdot g = \mathfrak{J}^2 g,\ \mathfrak{A} \cdot g = \mathfrak{A}^2 g$.

Remark E.2. $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ preserve the wavefront gauge and, consequently, induce an action on $\Phi = (e, \gamma, w)$. We illustrate this important fact by three examples. First,

$$\begin{aligned}(\mathfrak{A} \cdot \mathbf{F})_3{}^{\mu} \tfrac{\partial}{\partial (x')^{\mu}} &= \mathfrak{A}^{-2}\, \mathbf{F}_3{}^{\mu}\ \tfrac{\partial}{\partial x^{\mu}} = \mathfrak{A}^{-2} \left(e_4 \tfrac{\partial}{\partial x^1} + e_5 \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^4} \right) \\ &= \mathfrak{A}^{-2} \left(e_4\, \mathfrak{A}^{-1} \tfrac{\partial}{\partial (x')^1} + e_5\, \mathfrak{A}^{-1} \tfrac{\partial}{\partial (x')^2} + \mathfrak{A}^{2} \tfrac{\partial}{\partial (x')^4} \right)\end{aligned}$$

compatible with the wavefront gauge. Necessarily, $(\mathfrak{A} \cdot e)_i = \mathfrak{A}^{-3}\, e_i$ for $i = 4, 5$. Second, abbreviating $F'_a = \mathfrak{A} \cdot F_a$, we have

$$\begin{aligned}(\mathfrak{A} \cdot w)_5 = (\mathfrak{A} \cdot W)\big(F'_3,\ F'_2,\ F'_3,\ F'_2\big) &= \mathfrak{A}^{-6}\ (\mathfrak{A} \cdot W)\big(F_3,\ F_2,\ F_3,\ F_2\big) \\ &= \mathfrak{A}^{-6}\ \mathfrak{A}^{2}\ W\big(F_3,\ F_2,\ F_3,\ F_2\big) = \mathfrak{A}^{-4}\ w_5 .\end{aligned}$$

Third, abbreviating $F'_a = \mathfrak{Z} \cdot F_a$

$$\begin{aligned}(\mathfrak{Z} \cdot \Gamma)(F'_1,\ F'_1,\ F'_2) &= (\mathfrak{Z} \cdot g)\big((\mathfrak{Z} \cdot \nabla)_{F'_1} F'_1, F'_2\big) = \zeta\, g\big(\nabla_{F_1} F_1, F_2\big) + F_1(\zeta) \\ (\mathfrak{Z} \cdot \Gamma)(F'_1,\ F'_3,\ F'_4) &= (\mathfrak{Z} \cdot g)\big((\mathfrak{Z} \cdot \nabla)_{F'_1} F'_3, F'_4\big) = \zeta\, g\big(\nabla_{F_1} F_3, F_4\big)\end{aligned}$$

which is consistent with the wave front gauge if and only if $(\mathfrak{Z} \cdot \gamma)_3 = \zeta \gamma_3 + \frac{1}{2} F_1(\zeta)$ and $(\mathfrak{Z} \cdot \gamma)_4 = \zeta^{-1} \gamma_4 + \frac{1}{2} F_2(\zeta^{-1})$. By direct calculation, the present definitions of $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ are seen to be equivalent to those in Section 3 (generalizations in the case of $\mathfrak{C}$ and $\mathfrak{Z}$).

Proposition E.1. *Let S be one of $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$. Then, separately:*

- $(\star)$ *and* $(\star\star)$ *are preserved (see, Definition 2.1).*
- $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$ *if and only if* $(S \cdot \mathfrak{t},\ S \cdot \mathfrak{u},\ S \cdot \mathfrak{v}) = 0$.
- $(t, u, v) = 0$ *if and only if* $(S \cdot t,\ S \cdot u,\ S \cdot v) = 0$.

In particular, S is a field symmetry *in the sense of Definition 3.1.*

Proof. Let Riem be the Riemann curvature tensor associated to g, considered as a 4-covariant tensor. For each S, there are complex functions κ_1, κ_2, κ_3, κ_4 and a constant $\Omega > 0$ such that $\kappa_1 \kappa_2 \Omega^2 = \kappa_3 \kappa_4 \Omega^2 = 1$ and

$$(S \cdot \mathbf{F}_a)^{\mu} \tfrac{\partial}{\partial (x')^{\mu}} = \kappa_a\ \mathbf{F}_a{}^{\mu} \tfrac{\partial}{\partial x^{\mu}} \qquad a = 1, 2, 3, 4$$
$$S \cdot \nabla = \nabla \qquad S \cdot g = \Omega^2 g \qquad S \cdot \mathrm{Riem} = \Omega^2\, \mathrm{Riem} \qquad S \cdot W = \Omega^2\, W$$

We abbreviate $F'_a = S \cdot F_a = \kappa_a\, F_a$. By the definition of T, U, V in Appendix B,

$$(S \cdot T)^{\mu}(F'_a, F'_b) \tfrac{\partial}{\partial (x')^{\mu}} = \kappa_a \kappa_b\ T^{\mu}(F_a, F_b) \tfrac{\partial}{\partial x^{\mu}} \tag{E.1}$$

$$\begin{aligned}(S \cdot U)(F'_a, F'_b, F'_c, F'_d) = \Omega^2 \kappa_a \kappa_b \kappa_c \kappa_d\ \Big(& U(F_a, F_b, F_c, F_d) \\ & + g(F_a, F_b)\, T^{\mu}(F_c, F_d) \tfrac{1}{\kappa_a} \tfrac{\partial \kappa_a}{\partial x^{\mu}} \Big)\end{aligned} \tag{E.2}$$

$$(S \cdot V)(F'_a, F'_b, F'_c) = \kappa_a \kappa_b \kappa_c\, V(F_a, F_b, F_c) \tag{E.3}$$

By the definition of $(\mathfrak{t},\mathfrak{u},\mathfrak{v})$ and (t,u,v) in Proposition D.1,

	$\mathfrak{C}$	$\mathfrak{Z}$	$\mathfrak{J}$	$\mathfrak{A}$
$\mathfrak{t}=0 \iff S\cdot\mathfrak{t}=0$ and $t=0 \iff S\cdot t=0$		true	true	true
$\mathfrak{u}=0 \iff S\cdot\mathfrak{u}=0$ and $t=0 \iff S\cdot t=0$	true		true	true
$\mathfrak{v}=0 \iff S\cdot\mathfrak{v}=0$ and $v=0 \iff S\cdot v=0$	true	true	true	true

where

- in the case of $\mathfrak{t}$ and t we use (E.1), observing that $\frac{\partial}{\partial(x')^\mu}$ is proportional to $\frac{\partial}{\partial x^\mu}$ for $\mu=1,2,3,4$ if S is one of $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$,
- in the case of $\mathfrak{u}$ and u we use (E.2), observing that $\frac{\partial\kappa_a}{\partial x^\mu}=0$ for $a,\mu=1,2,3,4$ if S is one of $\mathfrak{C}$, $\mathfrak{J}$, $\mathfrak{A}$,
- in the case of $\mathfrak{v}$ and v we use (E.3) and the transformation law for Λ.

The remaining cases are discussed separately. *Case 1:* If $S=\mathfrak{C}$, we have

$$\frac{\partial}{\partial x^a}=\frac{\partial\mathfrak{C}^1}{\partial x^a}\frac{\partial}{\partial(x')^1}+\frac{\partial\mathfrak{C}^2}{\partial x^a}\frac{\partial}{\partial(x')^2}\qquad(a=1,2)\qquad\qquad\frac{\partial}{\partial x^3}=\frac{\partial}{\partial(x')^3}$$

such that $\mathfrak{t}=0\iff\mathfrak{C}\cdot\mathfrak{t}=0$ and $t=0\iff\mathfrak{C}\cdot t=0$ follow from Proposition D.1. *Case 2:* If $S=\mathfrak{Z}$, note that for all a, b, c, d,

$$g(F_a,F_b)\,T^\mu(F_c,F_d)\tfrac{1}{\kappa_a}\tfrac{\partial\kappa_a}{\partial x^\mu}=g(F_a,F_b)\,\textstyle\sum_{i=1,2}T^i(F_c,F_d)\tfrac{1}{\kappa_a}\tfrac{\partial\kappa_a}{\partial x^i}$$

by the structure of the torsion-matrix (last column vanishes) and $\frac{\partial\kappa_a}{\partial x^3}=0$. Because of $\kappa_3=\kappa_4=1$ and the term $g(F_a,F_b)$, the expression vanishes unless $(a,b)=(1,2),\ (2,1)$. At this point, one verifies directly that

$$\begin{aligned}(\mathfrak{t},\mathfrak{u})=0\quad&\Longrightarrow\quad\mathfrak{Z}\cdot\mathfrak{u}=0, &\qquad (\mathfrak{Z}\cdot\mathfrak{t},\mathfrak{Z}\cdot\mathfrak{u})=0\quad&\Longrightarrow\quad\mathfrak{u}=0,\\ (t,u)=0\quad&\Longrightarrow\quad\mathfrak{Z}\cdot u=0, &\qquad (\mathfrak{Z}\cdot t,\mathfrak{Z}\cdot u)=0\quad&\Longrightarrow\quad u=0\end{aligned}$$

This concludes the proof. $\square$

F. An Estimate for Pole-Flip

Lemma F.1. *Let $B\subset\mathbb{R}^2$ be open. For all $0<r_1<r_2$ let*

$$A(r_1,r_2)=\big\{(\xi,\underline{u},u)\in\mathbb{R}^4\ :\ (\underline{u},u)\in B,\ r_1<|\xi|<r_2\big\}.$$

For all $|\alpha|\geq 1$, all integers $R\geq 0$, and all C^R-fields $\Phi:A(|\alpha|r_1,|\alpha|r_2)\to\mathcal{R}$,

$$\big\|\mathbf{Flip}_\alpha\cdot\Phi\big\|_{C^R(A(\frac{|\alpha|}{r_2},\,\frac{|\alpha|}{r_1}))}\ \lesssim_{(R,r_1,r_2)}\ \big\|\Phi\big\|_{C^R(A(|\alpha|r_1,\,|\alpha|r_2))}\tag{F.1}$$

For $\mathbf{Flip}_\alpha$, see Definition 3.6. The same estimate holds for the C^R norms on the image of the sets $A(\frac{|\alpha|}{r_2},\,\frac{|\alpha|}{r_1})$ and $A(|\alpha|r_1,\,|\alpha|r_2)$ under the change of coordinates from $x=(\xi,\underline{u},u)$ to $q=(t,\xi,\underline{u})$, see Convention 7.1.

Remark F.1. The point here is that the constant in (F.1) is independent of $|\alpha|\geq 1$ and B. Lemma F.1 is used in Step 9 of the proof of Theorem 8.1. If Theorem 8.1 would be stated with the additional condition $a=\mathfrak{A}$, then Lemma F.1 would not appear in its proof.

Proof. Without loss of generality, we can assume $\alpha \geq 1$, because $\mathbf{Flip}_\alpha = \mathbf{Flip}_{-\alpha}$. Let $\mathfrak{C}_\alpha$ be the angular coordinate transformation (Definition 3.2) given by $\mathfrak{C}(\xi) = \alpha\xi$. Let $\mathfrak{F}$ be the angular coordinate transformation with $\mathfrak{C}(\xi) = \xi^{-1}$. Let $\mathfrak{Z}$ be the $U(1)$ transformation (Definition 3.3) with $\zeta(\xi) = -\xi/\overline{\xi}$. We have $\mathbf{Flip}_\alpha = \mathfrak{Z}\circ\mathfrak{C}_\alpha\circ\mathfrak{F}\circ\mathfrak{C}_{1/\alpha} = \mathfrak{C}_\alpha \circ \mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{1/\alpha}$. Decompose $\Phi = (e, \gamma, w) = \Phi_1 \oplus \Phi_2$ where $\Phi_1 = (e_1, e_2, e_4, e_5)$ and $\Phi_2 = (e_3, \gamma_1, \ldots, \gamma_8, w_1, \ldots, w_5)$. Introduce the notation $\epsilon(1) = 1$ and $\epsilon(2) = 0$. For all $\beta = (\beta_1, \beta_2, 0, 0) \in \mathbb{N}_0^4$ with $|\beta| \leq R$, all $0 < a < b$, all $\lambda > 0$, all $0 \leq r \leq R$, all $N = 1, 2$, and all fields Φ,

$$\begin{aligned}
\|\partial^\beta(\mathfrak{C}_\lambda \cdot \Phi)_N\|_{C^0(A(\lambda a,\lambda b))} &= \lambda^{\epsilon(N)-|\beta|}\,\|\partial^\beta \Phi_N\|_{C^0(A(a,b))} \\
\|(\mathfrak{C}_\lambda \cdot \Phi)_N\|_{C^r(A(\lambda a,\lambda b))} &\leq \lambda^{\epsilon(N)-r}\,\|\Phi_N\|_{C^r(A(a,b))} \quad \text{for} \quad 0 < \lambda \leq 1 \\
\|(\mathfrak{F} \cdot \Phi)_N\|_{C^r(A(b^{-1},a^{-1}))} &\lesssim_{(R,a,b)} \|\Phi_N\|_{C^r(A(a,b))} \\
\|(\mathfrak{Z} \cdot \Phi)_1\|_{C^r(A(a,b))} &\lesssim_{(R,a,b)} \|\Phi_1\|_{C^r(A(a,b))} \\
\|(\mathfrak{Z} \cdot \Phi)_2\|_{C^r(A(a,b))} &\lesssim_{(R,a,b)} \|\Phi_1\|_{C^r(A(a,b))} + \|\Phi_2\|_{C^r(A(a,b))}
\end{aligned}$$

Set $X = (R, r_1, r_2)$. The above estimates imply

$$\begin{aligned}
&\|\partial^\beta(\mathbf{Flip}_\alpha \cdot \Phi)_2\|_{C^0(A(\frac{\alpha}{r_2},\frac{\alpha}{r_1}))} \\
&= \|\partial^\beta((\mathfrak{C}_\alpha \circ \mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^0(A(\frac{\alpha}{r_2},\frac{\alpha}{r_1}))} \\
&= \alpha^{-|\beta|}\|\partial^\beta((\mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^0(A(\frac{1}{r_2},\frac{1}{r_1}))} \\
&\lesssim_X \alpha^{-|\beta|}\|((\mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_1\|_{C^{|\beta|}(A(\frac{1}{r_2},\frac{1}{r_1}))} + \alpha^{-|\beta|}\|((\mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^{|\beta|}(A(\frac{1}{r_2},\frac{1}{r_1}))} \\
&\lesssim_X \alpha^{-|\beta|}\|(\mathfrak{C}_{\frac{1}{\alpha}} \cdot \Phi)_1\|_{C^{|\beta|}(A(r_1,r_2))} + \alpha^{-|\beta|}\|(\mathfrak{C}_{\frac{1}{\alpha}} \cdot \Phi)_2\|_{C^{|\beta|}(A(r_1,r_2))} \\
&\lesssim_X \alpha^{-1}\|\Phi_1\|_{C^{|\beta|}(A(\alpha r_1,\alpha r_2))} + \|\Phi_2\|_{C^{|\beta|}(A(\alpha r_1,\alpha r_2))} \\
&\lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1,\alpha r_2))}
\end{aligned}$$

Similarly, $\|\partial^\beta(\mathbf{Flip}_\alpha \cdot \Phi)_1\|_{C^0(A(\frac{\alpha}{r_2},\frac{\alpha}{r_1}))} \lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1,\alpha r_2))}$. Together,

$$\|\partial^\beta(\mathbf{Flip}_\alpha \cdot \Phi)\|_{C^0(A(\frac{\alpha}{r_2},\frac{\alpha}{r_1}))} \lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1,\alpha r_2))}$$

The last estimate, and the fact that $\frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^4}$ both commute with $\mathbf{Flip}_\alpha$, imply Lemma F.1. □

G. Supplement to Proposition 8.1.

Here we make explicit all the polynomials $\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ (see, Definition 8.1) that appear in Proposition 8.1. In particular, it will be clear, by inspection, that these polynomials are independent of K, as required. We use the $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block-notation, as in **(S6)** and **(S7)**, and the complex conjugation operator C.

- Recall that, for each μ, the square matrices $B_1^\mu(q, 0)$, $B_2^\mu(q, 0)$ and $B_3^\mu(q, 0)$ are 5×5, 9×9 and 4×4, respectively. See **(S6)**.

In equation (8.5a), we have $B_i^\mu(q,0) - \not{B}_i^\mu = u^{-2}\mathcal{G}_K$ (that is, $\boldsymbol{\mathcal{H}} = 0$) for all $i = 1,2,3$ and $\mu = 0,1,2,3$, except in the cases $(i,\mu) = (1,A)$ with $A = 1,2$, when

$$B_1^1(q,0) - \not{B}_1^1 = \tfrac{1}{u}\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&-\mathbf{e} \\ 0&0&0&-\mathbf{e}&0 \end{pmatrix} + \tfrac{1}{u^2}\mathcal{G}_K = \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K$$

and similar for $B_1^2(q,0) - \not{B}_1^2$.

- For (8.5b),

$$\Big(Q_{11} - \not{Q}_{11}\Big|\, Q_{12} - \not{Q}_{12}\Big|\, Q_{13} - \not{Q}_{13}\Big) - \tfrac{1}{u^2}\mathcal{G}_K$$
$$= \tfrac{1}{u}\left(\begin{array}{ccccc|ccc|cccc} 0&0&0&0&0&0&\cdots&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdots&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdots&0&0&0&0&0 \\ 0&0&0&0&4\boldsymbol{\lambda}&0&\cdots&0&0&0&0&0 \\ 0&0&-3\,\omega_7(0)&-2\overline{\boldsymbol{\lambda}}&\mathfrak{A}^2&0&\cdots&0&0&0&0&-3\,z_3(0) \end{array}\right)$$

- For (8.5c),

$$\Big(Q_{21} - \not{Q}_{21}\Big|\, Q_{22} - \not{Q}_{22}\Big|\, Q_{23} - \not{Q}_{23}\Big) - \tfrac{1}{u}\mathcal{G}_K$$
$$= \left(\begin{array}{cc|cccccccc|cccc} 0&\cdots&\cdots&0&\mathbf{e}&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&-i\mathbf{e}&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&0&0&-\mathbf{e}\,\mathfrak{C}_+&\mathbf{e}\,\mathfrak{C}_+&0&0&-\mathbf{e}\,\mathfrak{C}_+&0&0 \\ 0&\cdots&\cdots&0&0&0&\mathbf{e}\,\mathfrak{C}_-&\mathbf{e}\,\mathfrak{C}_-&0&0&-\mathbf{e}\,\mathfrak{C}_-&0&0 \\ 0&\cdots&\cdots&0&0&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&-\mathfrak{C}_+\,\overline{\omega_1(0)}&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&-\overline{\boldsymbol{\lambda}}&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&-\boldsymbol{\lambda}\,C&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&0&0&\mathfrak{C}_-\,\overline{\boldsymbol{\lambda}}&\mathfrak{C}_-\,\boldsymbol{\lambda}&0&0&\mathfrak{C}_-\,\boldsymbol{\lambda}&0&0 \end{array}\right)$$

where the operators $\mathfrak{C}_+\,\varphi$ and $\mathfrak{C}_-\,\varphi$ are defined by $\varphi + \overline{\varphi}\,C$ and $i(\varphi - \overline{\varphi}\,C)$, respectively, where φ is any complex valued function.

- For (8.5d)

$$\Big(Q_{31} - \not{Q}_{31}\Big|\, Q_{32} - \not{Q}_{32}\Big|\, Q_{33} - \not{Q}_{33}\Big) - \tfrac{1}{u^2}\mathcal{G}_K$$
$$= \big(\tfrac{1}{|t|} + \tfrac{1}{u}\big)\left(\begin{array}{cc|ccccc|cccc} 0&\cdots&\cdots&0&0&0&1+C&2&0&0&0 \\ 0&\cdots&\cdots&0&-C&1&0&0&1&0&0 \\ 0&\cdots&\cdots&0&0&0&0&0&0&0&0 \\ 0&\cdots&\cdots&0&0&0&0&0&0&0&0 \end{array}\right)$$

- For (8.7b), (8.7c), (8.7d), recall from Remark 5.3 that $\mathbf{f}(q,\Psi)$ is a quadratic polynomial in Ψ, $\overline{\Psi}$ without constant term. Let $\mathbf{f}(q,\Psi) = \mathbf{f}_{(1)}(q,\Psi) + \mathbf{f}_{(2)}(q,\Psi)$ be its decomposition into homogeneous (over $\mathbb{R}$) parts. By definition (8.3c),

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} Q(q, s\Xi)\,\Pi = \tfrac{\mathrm{d}^2}{\mathrm{d}s_1\,\mathrm{d}s_2}\Big|_{(s_1,s_2)=0}\ \tfrac{1}{2}\,\pi^{-1}\,\mathbf{f}_{(2)}\big(q,\ \pi(s_1\Xi + s_2\Pi)\big)$$

It follows from direct inspection of (5.8b) that $\mathbf{f}_{(2)}(q,\Psi)$ is a polynomial in Ψ, $\overline{\Psi}$ whose coefficients are Laurent polynomials in $\frac{1}{u}$, with complex coefficients. Now one reads off from (5.10) that the Laurent polynomials have the structure recorded in (8.7b), (8.7c) and (8.7d).

- For (8.7a), recall from Remark 5.3 that $\mathbf{A}(q,\Psi)$ is affine linear (over $\mathbb{R}$) in Ψ. Let $\mathbf{A}^\mu(q,\Psi) = \mathbf{A}^\mu_{(0)}(q,\Psi) + \mathbf{A}^\mu_{(1)}(q,\Psi)$ be its decomposition into homogeneous parts. By (8.3b), we have $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathbf{B}^\mu(q,s\Xi) = \pi^{-1}\,\mathbf{A}^\mu_{(1)}(q,\pi\,\Xi)\pi$. Writing out the result in the notation $(h,\sigma,\ell) = \pi\,(\Xi_1,\Xi_2,\Xi_3)$ of (8.2), we obtain for $\mu = 0$ and $\mu = 3$,

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathbf{B}^\mu(q,s\Xi) = \mathrm{diag}\big(0,\ \tfrac{1}{u^4}h_3,\ \tfrac{1}{u^4}h_3,\ \tfrac{1}{u^4}h_3,\ \tfrac{1}{u^2}h_3\big) \oplus \big(\tfrac{1}{u^2}h_3\,\mathbb{1}_9\big) \oplus 0_{4\times 4}$$

and for $\mu = A = 1,2$,

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathbf{B}^A(q,s\Xi)\\
&\quad= \begin{pmatrix}
\frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0 & 0 & 0\\
\frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0 & 0\\
0 & \frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0\\
0 & 0 & \frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^2}h_A\\
0 & 0 & 0 & \frac{1}{u^2}\overline{h}_A & 0
\end{pmatrix} \oplus 0_{9\times 9} \oplus \big(\tfrac{1}{u^3}h_{A+3}\,\mathbb{1}_4\big)
\end{aligned}$$

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